COMPACT OPERATORS BETWEEN CERTAIN SEQUENCE SPACES

by

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APPROVAL PAGE

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ABSTRACT

In this study, the general theory of FK and BK spaces, the most important basic properties of certain measures of noncompactness on bounded sets of complete metric spaces and Banach spaces and some Hausdorff measure of noncompactness of operators between Banach spaces are examined. Also, some classes of linear operators and compact operators between Banach spaces are characterized.

Keywords: Sequence Spaces, FK and BK Spaces, Measures of Noncompactness, Bounded Linear Operators, Compact Operators.

BAZI DİZİ UZAYLARI ARASINDAKİ KOMPACT OPERATÖRLER

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ÖZ

Bu çalışmada FK ve BK uzaylarının genel teorisi, tam metrik ve Banach uzaylarının sınırlı cümleleri üzerinde kompaksızlığın belirli ölçümlerinin en önemli temel özellikleri ve operatörlerin kompaksızlığının Hausdorff ölçümü incelendi. Ayrıca, Banach uzayları arasında lineer ve compact operatörler karakterize edildi.

Anahtar Kelimeler: Dizi Uzayları, FK ve BK Uzayları, Kompaksızlığın Ölçümü, Sınırlı Lineer Operatörler, Kompak Operatörler.

To my parent *Dursun* & *Fethiye AYDIN*

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LIST OF SYMBOLS AND ABBREVIATIONS

SYMBOL/ABBREVIATION

ω	Set of all sequences with complex entries
c	Set of convergent sequences
c_0	Set of null sequences
ℓ_∞	Set of bounded sequences
ℓ_p	Set of p absolutely summable sequences
cs	Set of convergent series
bs	Set of bounded series
χ	(Function of) Hausdorff measure of noncompactness
B_X	Open unit ball in X
S_X	Unit sphere in X
$\overline{B}_X$	Closed unit ball in X
$B_r(x)$	An open ball of radius r, centered at x
$\overline{B}_r(x)$	An closed ball of radius r, centered at x
$\mathcal{M}_X$	All nonempty and bounded subsets in X
$\mathcal{M}_X^c$	The subfamily of $\mathcal{M}_X$ consisting of all closed sets
(X, Y)	Set of linear operators from X to Y
$\mathcal{L}(X, Y)$	Set of linear operators from a linear space X into a linear space Y
$\mathcal{C}(X, Y)$	Set of compact operators from X to Y
$\mathcal{B}(X, Y)$	Set of bounded linear operators from X to Y
$\mathbb{N}_0$	Set of natural numbers, that is, $\mathbb{N}_0 = \{0, 1, 2, ...\}$
$\mathbb{R}^+$	Set of non-negative real numbers
$\mathbb{R}$	Set of real numbers, the real field

$\mathbb{C}$	Set of complex numbers, the complex field
$\mathcal{F}$	Collection of all finite subsets of $\mathbb{N}$
$\mathbb{Q}$	Set of rational numbers
θ	Zero vector in a linear space X
ϕ	Set of all finitely non-zero sequences
$\emptyset$	Empty set
ℓ_1	Space of absolutely summable sequences
$x^{[m]}$	m^{th} section of a sequence $x = (x_k)$
$e^{(k)}$	Sequences whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$
Ax	$\{(Ax)_n\}_{n=0}^{\infty}$
$\{(Ax)_n\}$	A-transform of a sequence x
(λ, μ)	Class of all matrices from a sequence space λ into a sequence space μ
λ^{α}	$\alpha-$dual of a sequence space λ
λ^{β}	$\beta-$dual of a sequence space λ
λ^{γ}	$\gamma-$dual of a sequence space λ
X'	Continuous dual of a sequence space X
X^f	$f-$dual of a sequence space X
λ_A	Domain of an infinite matrix A in a sequence space λ
$\sum_k$	$\sum_{k=0}^{\infty}$

CHAPTER 1

INTRODUCTION

The theory of FK space was initiated by K. Zeller in 1949. K. Zeller published some seminar paper, for example, [5], [6], [8] around 1949. The subject was then further developed by Zeller and many other mathematicians.

Measures of noncompactness are very useful tools in functional analysis, for example in metric fixed point theory and in the theory of operator equations in Banach spaces. They are very often used in the studies of functional equations, ordinary and partial differential equations, optimal control theory, etc. In particular, the characterizations of compact operators between Banach spaces can be obtained from them.

The first measure of noncompactness, denoted by α, was defined and studied by Kuratowski in 1930. Other measures of noncompactness have been defined since then. The most important one of them is the Hausdorff measure of noncompactness χ which was introduced by Goldenštein, Go'hberg and Markus in 1957.

We give an introduction to the general theory of FK spaces as well as BK, AK and AD spaces, an axiomatic introduction of measures of noncompactness on bounded sets of complete metric spaces, and study their most important properties. In particular, we consider the Kuratowski and Hausdorff measures of noncompactness. Moreover, we study the Hausdorff measures of noncompactness of operators between Banach spaces. Lastly, we establish an identity for the Hausdorff measure of noncompactness of bounded sets in the space ℓ_1 of all absolutely convergent series of complex numbers and to characterize some classes of all compact bounded operators (ℓ_1, ℓ_1), $\left((\ell_p)_{\bar{N}_q}, c_0\right)$, $\left((\ell_p)_{\bar{N}_q}, c\right)$ and $\left((\ell_p)_{\bar{N}_q}, \ell_1\right)$.

1.1 PRELIMINARIES

The concept of a linear space involves an algebraic structure given by the definition of two operations, namely the sum of any two of its vectors and the product of any scalar with any vector. On the other hand a topological structure of a set may be given by a metric.

Definition 1.1. *Let $X \neq \emptyset$ be a set. A function*

$$d : X \times X \to \mathbb{R}$$

is said to be a metric *for X if the following condition are satisfied for all $x, y, z \in X$*

$(M1)$ $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$

$(M2)$ $d(x, y) = d(y, x)$ *(symmetry)*

$(M3)$ $d(x, y) \leq d(x, z) + d(z, y)$ *(triangle inequality).*

The set X together with a metric d is called a metric space *denoted by (X,d).*

If a set is both a linear and metric space, then it will be natural to require the algebraic operations to be continuous with respect to the metric. The continuity of the algebraic operations of a linear metric space (X, d) means the following: If (x_n) and (y_n) are two sequences in X and (λ_n) is a sequence of scalars with $x_n \to x$, $y_n \to y$ and $\lambda_n \to \lambda$ $(n \to \infty)$, then $x_n + y_n \to x + y$ and $\lambda_n x_n \to \lambda x$ $(n \to \infty)$. This means that $d(x_n, x)$, $d(y_n, y) \to 0$ and $\lambda_n \to \lambda$ $(n \to \infty)$ together imply $d(x_n + y_n, x + y) \to 0$ and $d(\lambda_n x_n, \lambda x) \to 0$ $(n \to \infty)$.

Definition 1.2. *Let X be a linear space and d a metric on X. Then (X, d), or X for short, is said to be a* linear metric space, *if the algebraic operations on X are continuous functions.*

Definition 1.3. *Let X be a metric sapce. Then X is said to be complete if every Cauchy sequence in X converges.*

Definition 1.4. *A complete linear metric space is said to be a Frechet space (cf. [11, Definition 5.3.2, p. 78]).*

The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value. The paranorm of a vector x may be thought of as the distance from x to the origin 0.

Definition 1.5. *Let X be a linear space. A function $p : X \to \mathbb{R}$ is called paranorm, if*

$(P.1)$ $p(0) = 0$

$(P.2)$ $p(x) \geq 0$ *for all* $x \in X$

$(P.3)$ $p(-x) = p(x)$ *for all* $x \in X$

$(P.4)$ $p(x + y) \leq p(x) + p(y)$ *for all* $x, y \in X$ *(triangle inequality)*

$(P.5)$ *if* (λ_n) *is a sequence of scalars with* $\lambda_n \to \lambda$ $(n \to \infty)$ *and* (x_n) *is a sequence of vectors with* $p(x_n - x) \to 0$ $(n \to \infty)$, *then* $p(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty)$ *(continuity of multiplication by scalars).*

If p is a paranorm on X, then (X, p), or X for short, is called a **paranormed space**. A paranorm p for which $p(x) = 0$ implies $x = 0$ is called **total**. For any two paranorms p and q, p is called **stronger** than q if, whenever (x_n) is a sequence such that $p(x_n) \to 0$ $(n \to \infty)$, then also $q(x_n) \to 0$ $(n \to \infty)$. If p is stronger than q, then q is said to be **weaker** than p. If p is stronger than q and q is stronger than p, then p and q are called **equivalent**. If p is stronger than q, but p and q are not equivalent, then p is said to be **strictly stronger** than q, and q is called **strictly weaker** than p.

It is easy to see that every totally paranormed space is a linear metric space. The converse is also true. The metric of any linear metric space is given by some total paranorm (cf. [11, Theorem 10.4.2, p. 183]).

A sequence of paranorms may be used to define a paranorm.

Theorem 1.1. *[16, Theorem 1.2] Let $(p_k)_{k=0}^{\infty}$ be a sequence of paranorms on a linear space X. We define the so-called Fŕechet combination of (p_k) by*

$$p(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{p_k(x)}{1 + p_k(x)} \tag{1.1}$$

Then

(a) p is a paranorm on X and satisfies

$$p(x_n) \to 0\,(n \to \infty) \text{ if and only if } p_k(x_n) \to 0\,(n \to \infty) \text{ for each } k\,; \tag{1.2}$$

(b) p is the weakest paranorm which is stronger than every p_k;

(c) p is total if and only if every p_k is total.

A subset S of a linear space X is said to be ***absorbing*** if for each $x \in X$ there is $\varepsilon > 0$ such that $\lambda x \in S$ for all scalars λ with $|\lambda| \leq \varepsilon$.

Remark 1.1. *Let (X, p) be a paranormed space. Then the open neighbourhoods of 0, $N_r(0) = \{x \in X : p(x) < r\}$, are absorbing for all $r > 0$.*

Proof. We assume that $N_r(0)$ is not absorbing for some $r > 0$. Then there are $x \in X$ and a sequence $\lambda = (\lambda_n)_{n=0}^{\infty}$ of scalars with $\lambda_n \to 0$ $(n \to \infty)$ and $\lambda x \notin N_r(0)$ for all $n = 0, 1, \ldots$ But this means $p(\lambda_n x) \geq r$ for all n. Let $\lambda_n \to 0$ and $x_n = x$. Hence, $p(\lambda_n x_n - \lambda x) = p(\lambda_n x)$ $(\star)$. Since $p(\lambda_n x) \geq r$ if we take $n \to \infty$ in $(\star)$, we observe multiplication by scalars is not continous. This is a contradiction with (P.5) of Definition 1.5. So the remark is valid. □

As a special case of Theorem 1.1, we obtain

Theorem 1.2. *The set ω is a Fréchet space with respect to the metric d defined by*

$$d(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|} \text{ for all } x, y \in \omega \tag{1.3}$$

Furthermore convergence in (ω, d) and coordinatewise convergence are equivalent, that is $x^{(n)} \to x$ $(n \to \infty)$ in (ω, d) if and only if $x_k^{(n)} \to x_k$ $(n \to \infty)$ for every k.

Now we introduce the concept of a *Schauder basis*.

Definition 1.6. ***A Schauder basis*** *of a linear metric space X is a sequence* (b_n) *of vectors such that for each vector* $x \in X$ *there is a unique sequence* (λ_n) *of scalars with* $\sum_{k=0}^{\infty} \lambda_n b_n = x$, *that is* $\lim_{m\to\infty} \sum_{k=0}^{m} \lambda_n b_n = x$.

For finite dimensional spaces, the concepts of Schauder and algebraic bases coincide. In most cases, however, the concepts differ. Every linear space has an algebraic basis. But there are linear metric spaces without a Schauder basis, as we shall see later in this subsection.

Example 1.1. *For each* $n = 0, 1, \ldots$, *let* $e^{(n)}$ *be the sequence with* $e_n^{(n)} = 1$ *and* $e_k^{(n)} = 0$ *for* $k \neq n$. *Then* $\left(e^{(n)}\right)_{n=0}^{\infty}$ *is a Schauder basis of* ω. *More precisely, every sequence* $x = (x_k)_{k=0}^{\infty} \in \omega$ *has a unique representation* $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, *that is,* $\lim_{m\to\infty} x^{[m]} = x$ *for* $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)}$, *the m-section of x.*

A metric space (X, d) is called ***separable*** if it has a *countable dense set.* That means there is a countable set $A \subset X$ such that for all $\varepsilon > 0$ and for all $x \in X$ there is an element $a \in A$ with $d(x, a) < \varepsilon$.

Definition 1.7. *Let X be a vector space. A real-valued function* $||.||$ *on X is called a norm on X if it has the following properties for arbitrary vectors* $x, y \in X$ *and any scalar* λ:

$(N1)$ $||x|| \geq 0$ *and* $||x|| = 0$ *if and only if* $x = 0$

$(N2)$ $||\lambda x|| = |\lambda| ||x||$ *(homogenity)*

$(N3)$ $||x + y|| \leq ||x|| + ||y||$ *(triangle inequality).*

A vector space X with a norm defined on it is called a normed space.

Definition 1.8. *A complete normed space is called a* Banach space.

Theorem 1.3. *Every complex linear metric space X with Schauder basis is separable.*

Proof. Let (b_n) be a Schauder basis of X. For each $m \in \mathbb{N}$, we put

$$A_m = \left\{ \sum_{n=1}^{m} \rho_n b_n : \rho_n \in \mathbb{Q} + i\mathbb{Q} \; (n = 1, 2, \ldots, m) \right\} \text{ and } A = \bigcup_{m=1}^{\infty} A_m$$

Then A is a countable set in X and now we need to show that A is dense in X. Since X has a Schauder basis, for any $x \in X$ there is a unique sequence (λ_n) of scalars with $\sum_{n=0}^{\infty} \lambda_n b_n = x$, that is $\lim_{m\to\infty} \sum_{n=0}^{m} \lambda_n b_n = x$ where $\lambda_n \in \mathbb{C}$ for all $n \in \mathbb{N}$. We can write $\lambda_n = \beta_n + i\gamma_n$ as $\beta_n, \gamma_n \in \mathbb{R}$. Take any $a \in X$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, that is, $\overline{\mathbb{Q}} = \mathbb{R}$, we can find $|\beta_n - \eta_n| < \varepsilon$ and $|\gamma_n - \zeta_n| < \varepsilon$ with $\eta_n, \zeta_n \in \mathbb{Q}$. We can also describe $\alpha_n = \eta_n + i\zeta_n$. So we write $|\lambda_n - \alpha_n| < 2\varepsilon$ $(\star\star)$. If we consider $d(x, a) < \varepsilon$ and $(\star\star)$ and if we define $a =: \sum_{n=0}^{\infty} \alpha_n b_n$, then $a \in A_\infty \in \bigcup_{m=1}^{\infty} A_m = A$. So A is dense in X. □

Example 1.2. *The set $\ell_\infty = \{x \in \omega : \sup_k |x_k| < \infty\}$ of all bounded sequences is a Banach space with $||x||_\infty = \sup_k |x_k|$ $(x \in \ell_\infty)$ which has no Schauder basis.*

Proof. It is well-known that$(\ell_\infty, \|.\|_\infty)$ is a Banach space.
If we show that ℓ_∞ is not separable and apply Theorem 1.3, then ℓ_∞ has no Schauder basis. We assume that ℓ_∞ is separable. Then there is a countable dense set $A = \{a_n : n = 0, 1, 2, \ldots\} \subset \ell_\infty$. For every n, let $U_n = N_{\frac{1}{3}}(a_n) = \left\{x \in \ell_\infty : ||x - a_n||_\infty < \frac{1}{3}\right\}$. Since $A \subset \ell_\infty$ is dense, $\ell_\infty \subset \bigcup_{n=0}^{\infty} U_n$. The set

$$B = \{x \in \omega : x_k \in \{0, 1\} \text{ for all k=0,1,}\ldots\} \subset \ell_\infty$$

is uncountable. Therefore there must be a set U_m which contains at least two distinct sequences x and $x^{'}$ of B. Then

$$||x - x^{'}||_\infty \geq 1 \text{ and } ||x - x^{'}||_\infty \leq ||x - a_m||_\infty + ||a_m - x^{'}||_\infty < 2/3,$$

a contradiction. Therefore ℓ_∞ cannot be separable. □

We introduce the so-called *classical sequence spaces*

$$\ell_\infty = \left\{x \in \omega : \sup_k |x_k| < \infty\right\},$$
$$c = \left\{x \in \omega : \lim_{k\to\infty} x_k = l \text{ for some } l \in \mathbb{C}\right\},$$
$$c_0 = \left\{x \in \omega : \lim_{k\to\infty} x_k = 0\right\}$$

of all *bounded*, *convergent* and *null sequences*, and

$$\ell_p = \left\{ x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty \right\} \text{ for } 1 \leq p < \infty.$$

The following result gives the algebraic and topological properties of the sets ℓ_∞, c, c_0 and ℓ_p.

Theorem 1.4. *[16, Theorem 1.10]*

(a) *Each of the sets ℓ_∞, c_0 and c is a Banach space with $||.||_\infty$ defined by $||x||_\infty = \sup_k |x_k|$. Moreover $|x_k| \leq ||x||_\infty$ for all $k = 0, 1, \ldots$.*

(b) *The sets ℓ_p are Banach spaces for $1 \leq p < \infty$ with $||.||_p$ defined by $||x||_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p}$. Moreover $|x_k| \leq ||x||_p$ for all $k = 0, 1, \ldots$.*

(c) *The sequence $\left(e^{(n)}\right)_{n=0}^{\infty}$ is a Schauder basis for each of the spaces c_0 and ℓ_p for $1 \leq p < \infty$. More precisely, every sequence $x = (x_n)_{n=0}^{\infty}$ in any of these spaces has a unique representation $x = \sum_{n=0}^{\infty} x_n e^{(n)}$.*

(d) *Let e be the sequence with $e_k = 1$ for all $k = 0, 1, \ldots$. We put $b^{(0)} = e$ and $b^{(n)} = e^{(n-1)}$ for $n = 1, 2, \ldots$. Then the sequence $(b^{(n)})_{n=0}^{\infty}$ is a Schauder basis for c. More precisely, every sequence $x = (x_n)_{n=0}^{\infty} \in c$ has a unique representation $x = le + \sum_{n=0}^{\infty}(x_n - l)e^{(n)}$ where $l = l(x) = \lim_{n\to\infty} x_n$.*

(e) *The space ℓ_∞ has no Schauder basis.*

If A is an infinite matrix with complex entries a_{nk} $(n, k \in \mathbb{N})$, then we write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^{\infty}$. Also, we write A_n for the sequence in the n-th row of A, that is, $A_n = (a_{nk})_{k=0}^{\infty}$ for every $n \in \mathbb{N}$. In addition, if $x = (x_k) \in \omega$, then we define the A-transform of x as the sequence $A(x) = (A_n(x))_{n=0}^{\infty}$ where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \ (x \in X) \text{ for all } n = 0, 1, \ldots \tag{1.4}$$

provided the series on the right converges for each $n \in \mathbb{N}$.

Definition 1.9. *Let X and Y be vector spaces over the same field. An operator $T : X \to Y$ is said to be linear if for all $x, y \in X$ and scalars λ,*

$$T(x + y) = Tx + Ty \tag{1.5}$$

$$T(\lambda x) = \lambda T x. \tag{1.6}$$

Using the notation Tx instead of $T(x)$ is a standard simplification. The *null space* of T, denoted by $N(T)$, is the set of all $x \in X$ such that $Tx = 0$.

The word "*kernel*" is also used for null space.

Definition 1.10. *Let X and Y be complete linear metric spaces and then a linear operator T from X to Y is called* compact or completely continuous *if $D(T) = X$ for the domain of T, and for every bounded sequence (x_n) in X, the sequence $(T(x_n))$ has a subsequence which converges in Y.*

Remark 1.2. *A compact operator is bounded, thus continuous.*

Definition 1.11. *For the sequence spaces λ and μ, the set $\mathcal{S}(\lambda, \mu)$ defined by*

$$\mathcal{S}(\lambda, \mu) \quad := \quad \{z = (z_k) \in w : \; xz = (x_k z_k) \in \mu \textit{ for all } x = (x_k) \in \lambda\} \tag{1.7}$$

is called the multiplier space of λ and μ. With the notation of (1.7), the alpha-, beta- and gamma-duals of a sequence space λ which are denoted by λ^α, λ^β and λ^γ, respectively, are defined by

$$\lambda^\alpha = \mathcal{S}(\lambda, \ell_1), \qquad \lambda^\beta = \mathcal{S}(\lambda, cs) \quad \textit{and} \quad \lambda^\gamma = \mathcal{S}(\lambda, bs),$$

that is

$$\lambda^\alpha = \{a = (a_k) \in \omega : a \cdot x = (a_k x_k)_{k=0}^\infty \in \ell_1 \textit{ for all } x \in X\},$$

$$\lambda^\beta = \{a = (a_k) \in \omega : a \cdot x = (a_k x_k)_{k=0}^\infty \in cs \textit{ for all } x \in X\},$$

and

$$\lambda^\gamma = \{a = (a_k) \in \omega : a \cdot x = (a_k x_k)_{k=0}^\infty \in bs \textit{ for all } x \in X\}.$$

Definition 1.12. *Let X be a vector space.*

(a) *A subset C of X is said to be* convex *if $\lambda x + \mu y \in C$ for all $x, y \in C$ and all scalars λ and μ with $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$. In other words, a subset C of X is said to be convex if $\lambda x + (1 - \lambda) y \in C$ for all $x, y \in C$ and for all $\lambda \in (0, 1)$.*

(b) *The* convex hull *of a subset S of X is the intersection of all convex sets that contain S; it is denoted by* co(S).

(c) *A* convex combination of elements *of a set S is an element of the form*

$$\sum_{k=1}^{n} \lambda_k x_k \text{ where } x_k \in S, \lambda_k \geq 0 \ (k = 1, ..., n) \text{ and } \sum_{k=1}^{n} \lambda_k = 1 \ (n \in \mathbb{N}).$$

The set of all convex combinations of elements of S is denoted by $cvx(S)$.

Now we state two fundamental results,

Theorem 1.5. *Let X be a linear space over $\mathbb{C}$ (or $\mathbb{R}$), and $C, C_1, \ldots, C_n$ be convex subsets of X and S be any subset of X. Then we have*

$$cvx(C) \subset C; \tag{1.8}$$

$$co(S) = cvx(S); \tag{1.9}$$

$$co\left(\bigcup_{k=1}^{n} C_k\right) = \left\{\sum_{k=1}^{n} \lambda_k C_k : \lambda_k \geq 0 \ (k = 1, \ldots, n), \ \sum_{k=1}^{n} \lambda_k = 1\right\} \tag{1.10}$$

Proof. (i)

We will prove (1.8) using the method of mathematical induction, it is enough to show that for any $n \geq 2$

$$x_k \in C, \ \lambda_k \geq 0 \ (k = 1, \ldots, n) \text{ and } \sum_{k=1}^{n} \lambda_k = 1 \text{ together imply } \sum_{k=1}^{n} \lambda_k x_k \in C \tag{1.11}$$

For $n = 2$, the statement is clearly true since C is a convex subset. Now we suppose that the statement in (1.11) is true for a natural number $n \geq 2$, and prove the statement for $n+1$. Let $x_k \in C$, $\lambda_k \geq 0$ for $k = 1, \ldots, n+1$ and $\sum_{k=0}^{n+1} \lambda_k = 1$, then there are two cases:

- If $\sum_{k=0}^{n} \lambda_k = 0$ then we have $\lambda_k = 0$ for $k = 1, \ldots, n$, $\lambda_{n+1} = 1$ and so $\sum_{k=0}^{n+1} \lambda_k x_k = x_{k+1} \in C$.

- If $\Lambda = \sum_{k=0}^{n} \lambda_k \neq 0$, then we have

$$\sum_{k=1}^{n+1} \lambda_k x_k = \Lambda \sum_{k=1}^{n} \frac{\lambda_k}{\Lambda} x_k + \lambda_{n+1} x_{n+1} \in C,$$

since

$$\eta_k = \frac{\lambda_k}{\Lambda} \geq 0 \ (k = 1, \ldots, n), \ \sum_{k=1}^{n} \eta_k = 1 \text{ and } y = \sum_{k=1}^{n} \eta_k x_k \in C \text{ by hypothesis .}$$

furthermore

$$\Lambda + \lambda_{n+1} = 1, \ \Lambda, \lambda_{n+1} \geq 0$$

imply $\Lambda y + \lambda_{n+1} x_{n+1} \in C$.

The inclusion in (1.11) is established.

(ii) Now we show (1.9).

It follows from (1.11) that

$$cvx(S) \subset co(S). \tag{1.12}$$

Since co(S) is a convex subset of X, it suffices to show that cvx(S) is convex. Let $\lambda \in (0,1)$ and $x, y \in cvx(S)$. Then there exist $n, m \in \mathbb{N}$, $\alpha_k \geq 0, x_k \in S$ for $k = 1, \ldots, n$ with $\sum_{k=1}^{n} \alpha_k = 1$, also $\beta_j \geq 0$, $y_j \in S$ for $j = 1, \ldots, m$ with $\sum_{j=1}^{m} \beta_j = 1$ such that

$$x = \sum_{k=1}^{n} \alpha_k x_k \text{ and } y = \sum_{j=1}^{m} \beta_j y_j.$$

Now

$$\sum_{k=1}^{n} \lambda \alpha_k + \sum_{j=1}^{m} (1-\lambda)\beta_j = \lambda + (1-\lambda) = 1$$

implies $\lambda x + (1-\lambda) y \in cvx(S)$. Hence we have proved (1.9).

(iii) Finally, we show (1.10).

We put

$$F = \left\{ \sum_{k=1}^{n} \lambda_k C_k : \lambda_k \geq 0 \ (k = 1, \ldots, n), \ \sum_{k=1}^{n} \lambda_k = 1 \right\}$$

It follows by (1.8) that

$$F \subset co\left(\bigcup_{k=1}^{n} C_k \right)$$

Since $\bigcup_{k=1}^{n} C_k \subset F$, it suffices to show that F is convex for the proof of (1.10). So let $\lambda \in (0,1)$ and $x, y \in F$. Now there exist $\lambda_k \geq 0$, $x_k \in C_k$ for $k = 1, \ldots, n$

with $\sum_{k=1}^{n} \alpha_k = 1$, and $\beta_j \geq 0$, $y_j \in C_j$ for $j = 1, \ldots, n$ with $\sum_{j=1}^{n} \beta_j = 1$ such that

$$x = \sum_{k=1}^{n} \alpha_k x_k \text{ and } y = \sum_{j=1}^{n} \beta_j y_j.$$

We put $\gamma_k = \lambda\alpha_k + (1-\lambda)\beta_k$ for $k = 1, \ldots, n$. Since the sets $C_1, ..., C_n$ are convex, there exist $z_k \in C_k$ for $k = 1, \ldots, n$ such that

$$\lambda\alpha_k x_k + (1-\lambda)\beta_k y_k = \gamma_k z_k \text{ for } k = 1, \ldots, n. \tag{1.13}$$

We observe that

$$\sum_{k=1}^{n} \gamma_k = \lambda \sum_{k=1}^{n} \alpha_k + (1-\lambda) \sum_{k=1}^{n} \beta_k = \lambda + (1-\lambda) = 1 \tag{1.14}$$

By (1.13) and (1.14), we have $\lambda x + (1-\lambda)y = \sum_{k=1}^{n} \gamma_k z_k \in F$. Thus (1.10) is proven.

□

Lemma 1.1. *Let X be a normed space and $Q \in \mathcal{M}_X$. Then we have for any $x \in X$*

$$\sup_{y \in co(Q)} ||x - y|| = \sup_{z \in Q} ||x - z|| \tag{1.15}$$

Proof. We know that $Q \subset co(Q)$, so $\sup_{z\in Q} ||x - z|| \leq \sup_{y \in co(Q)} ||x - y||$. It is clearly enough to show

$$\sup_{y \in co(Q)} ||x - y|| \leq \sup_{z \in Q} ||x - z|| \tag{1.16}$$

for the equation of (1.15). Let $y \in co(Q)$. Then there exist $x_k \in Q$ and $\lambda_k \geq 0$ for $k = 1, \ldots, n$ such that $\sum_{k=1}^{n} \alpha_k = 1$ and $y = \sum_{k=1}^{n} \alpha_k x_k$. It follows from

$$\begin{aligned} x - y &= x \sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{n} \alpha_k x_k \\ &= \sum_{k=1}^{n} \alpha_k x - \sum_{k=1}^{n} \alpha_k x_k = \sum_{k=1}^{n} \alpha_k (x - x_k), \end{aligned}$$

that

$$||x - y|| \leq \sum_{k=1}^{n} \alpha_k ||x - x_k|| \leq \sup_{x_k \in Q} ||x - x_k|| = \sup_{z \in Q} ||x - z||.$$

This implies (1.16). □

Definition 1.13. *Let Q be a nonempty and bounded subset of a normed space X. Then the* convex closure *of Q is the smallest convex and closed subset of X that contains Q, and denoted $Conv(Q)$.*

Remark 1.3. *It is easy to show that*

$$Conv(Q) = \overline{co(Q)}. \tag{1.17}$$

Proof. Since $Conv(Q)$ both contains Q and is smallest closed convex, $Conv(Q) \subset \overline{co(Q)}$. Let $a \in \overline{co(Q)}$.
Since $\overline{co(Q)} = \bigcap \{C_i : i \in I,\ Q \subset C_i,\ C_i$ is closed and convex, I is an index set $\}$, $a \in Conv(Q)$. So $\overline{co(Q)} \subset Conv(Q)$. □

Definition 1.14. *The* diameter of a set *S in a metric space (X, d) is the number* $\sup\{d(x,y) : x, y \in S\}$*; it is denoted by $diam(S)$.*

Some important properties of the *diameter* are the following [17]*:*

Lemma 1.2. (i) *$diam(S) = 0$ if and only if S is an empty set or consists of exactly one point.*

(ii) *If $S_1 \subset S_2$, then $diam(S_1) \leq diam(S_2)$;*

(iii) *$diam(\overline{S}) = diam(S)$.*

(iv) Cantor's Intersection Theorem*: If S_n is a decreasing sequence of nonempty, closed and bounded subsets of X and* $\lim_{n\to\infty} diam(S_n) = 0$*, then the intersection S_∞ of all S_n is nonempty and consists of exactly one point.*

Moreover, if X is a Banach space, then:

(v) *$diam(cS) = |c| diam(S)$ for any scalar c,*

(vi) *$diam(x + S) = diam(S)$ for any $x \in X$,*

(vii) *$diam(S_1 + S_2) \leq diam(S_1) + diam(S_2)$,*

$(viii)$ *$diam(co(S)) = diam(S)$.*

Proof. The parts $(i)-(vii)$, except (iv), can be shown in a direct way and (iv) is known from elementary functional analysis, and $(viii)$ is an immediate consequence of Lemma 1.1. □

Corollary 1.1. *Let X be a normed space and $Q \in \mathcal{M}_X$. Then, by Definition 1.14 and Lemma 1.2, we have*

$$diam(Q) = diam(Conv(Q)). \tag{1.18}$$

Definition 1.15. *Let X be a metric space. If M and S are subsets of X and $\varepsilon > 0$, then the set S is called $\varepsilon - net$ of M if for any $x \in M$ there exists $s \in S$, such that $d(x,s) < \varepsilon$.*

Equivalently,

$$M \subset S + \varepsilon\overline{B}_1(0) = \left\{s + \varepsilon b : s \in S, b \in \overline{B}_1(0)\right\}. \tag{1.19}$$

If the set S is finite, then the $\varepsilon - net$ of M is called finite *$\varepsilon - net$.*

A subset M of a metric space X is compact if every sequence (x_n) in M has a convergent subsequence, and in this case the limit of that subsequence is in M.

Definition 1.16. *The set M is said to be* relatively compact *if the closure $\overline{M}$ of M is a compact set. The set M is said to be totally bounded if it has a finite ε-net for every $\varepsilon > 0$.*

Definition 1.17. *The set M is said to be* totally bounded *if M is relatively compact.*

Remark 1.4. *If the metric space (X, d) is complete, then the set M is relatively compact if and only if it is totally bounded.*

CHAPTER 2

FK AND BK SPACES

In this chapter, we shall give a short introduction into the general theory of FK spaces and apply the results to characterize matrix transformations between the classical sequence spaces.

2.1 INTRODUCTION INTO THE THEORY OF FK SPACES

In this section, we shall give an introduction into the general theory of FK spaces. It is the most powerful tool for the solution of problems of various kinds in summability, in particular in the characterization of matrix transformations between sequence spaces. Most of the results of this subsection can be found in [23].

We saw in Theorem 1.2 that the set ω is a Fréchet space with the metric d defined in (1.3) and that convergence in ω and coordinatewise convergence are equivalent. Furthermore, by Theorem 1.4, the spaces ℓ_∞, c_0, c and ℓ_p $(1 \leq p < \infty)$ are Banach spaces with the norms $||.||_\infty$ and $||.||_p$, and convergence in any one of these spaces implies coordinatewise convergence by the inequalities in Theorem 1.4 parts (a) and (b). Thus the metric generated by these norms is stronger than the metric of ω on them.

Definition 2.1. *A Fréchet sequence space (X, d_X) is said to be an FK space if its metric d_X is stronger than the metric $d|_X$ of ω on X. A BK space is an FK space which is a Banach space.*

Remark 2.1. *[16, Remark 1.12] By definition, an FK space X is continuously embedded in ω, that is the* inclusion map $\iota : (X, d_X) \to (\omega, d)$ *defined by* $\iota(x) = x$ $(x \in X)$ *is continuous. An FK space X is a Fréchet sequence space with continuous coordinates $P_k : X \to \mathbb{C}$ defined by $P_k(x) = x_k$ $(k = 0, 1, ...)$ for all $x \in X$.*

Example 2.1. *The space ω is an FK space with its natural metric d. The spaces ℓ_∞, c_0, c and ℓ_p $(1 \leq p < \infty)$ are BK spaces with their natural norms.*

Theorem 2.1. *Let (X, d_X) be a Fréchet space, (Y, d_Y) an FK space and $f : X \to Y$ a linear map. Then $f : (X, d_X) \to (Y, d|_Y)$ is continuous if and only if $f : (X, d_X) \to (Y, d_Y)$ is continuous.*

Proof. Since f is linear, it is sufficient to show that f is continuous at 0 also $f(0) = 0$. First we assume that $f : (X, d_X) \to (Y, d_Y)$ is continuous. Let (x_n) be a sequence with $d(x_n, 0) \to 0$ $(n \to \infty)$. Then $d_Y(f(x_n), 0) \to 0$ $(n \to \infty)$, since d_Y is stronger than $d|_Y$ it follows that $d|_Y(f(x_n), 0) \to 0$ $(n \to \infty)$. So $f : (X, d_X) \to (Y, d|_Y)$ is continuous. Conversely we assume that $f : (X, d_X) \to (Y, d|_Y)$ is continuous. $(Y, d|_Y)$ and (X, d_X) are Hausdorff spaces because every metric space is a Hausdorff. Since $(Y, d|_Y)$ is a Hausdorff space and f is continuous, the *graph* of f, graph(f) $= \{(x, f(x)) : x \in X\}$, is a closed set in $(X, d_X) \times (Y, d|_Y)$ by the *closed graph lemma* (see appendix 8.4), hence the *graph* of f is a closed set in $(X, d_X) \times (Y, d_Y)$, since the FK metric d_Y is stronger than $d|_Y$. By the *closed graph theorem* (see appendix 8.5), the map $f : (X, d_X) \to (Y, d_Y)$ is continuous. □

Corollary 2.1. *Let X be a Fréchet space, Y an FK space, $f : X \to Y$ a linear map and P_n the n-th co-ordinate, that is, $P_n(y) = y_n$ $(y \in Y)$ for all $n = 0, 1, \ldots$ If each map $P_n \, o \, f : X \to \mathbb{C}$ is continuous, so is $f : X \to Y$.*

Proof. Since $P_n \, o \, f : X \to \mathbb{C}$ is continuous for each n, the map $f : X \to \omega$ is continuous by the equivalence of coordinatewise convergence and convergence in ω. Here, $Y \subset \omega$ and the metric on Y is the $d|_Y$ of ω on Y, so $f : (X, d_X) \to (Y, d|_Y)$ is continuous, hence $f : X \to Y$ is continuous by Theorem 2.1. □

We shall give the following result.

Remark 2.2. *Let $X \supset \phi$ be an FK space and $a \in \omega$. If the series $\sum_{k=0}^{\infty} a_k x_k$ converges for each $x \in X$, that is, $a \in X^\beta$ then the linear functional $f_a : X \to \mathbb{C}$ defined by*

$$f_a(x) = \sum_{k=0}^{\infty} a_k x_k \text{ for all } x \in X$$

is continuous.

Proof. We define the linear functional $f_{a,n} : X \to \mathbb{C}$ by $f_{a,n} = \sum_{k=0}^{n} a_k x_k$ for all $x \in X$ and for each $n \in \mathbb{N}_0$. Since X is an FK space, the coordinates $P_k : X \to \mathbb{C}$ are continuous on X for all $k = 0, 1, \ldots$, and so the functionals $f_{a,n} = \sum_{k=0}^{n} a_k P_k$ $(n = 0, 1, \ldots)$ are continuous. Also $f_a(x) = \lim_{n\to\infty} f_{a,n}(x)$ exists for each $x \in X$, and so $f_a : X \to \mathbb{C}$ is continuous by the Banach-Steinhaus theorem (see appendix 8.6). □

Theorem 2.2. *Any matrix map between FK spaces is continuous.*

Proof. Let X and Y be FK spaces, $A \in (X, Y)$ and the map $f_A : X \to Y$ be defined by $f_A(x) = A(x)$ for all $x \in X$. Since the maps $P_n \, o \, f_A : X \to \mathbb{C}$ are continuous with $(P_n \, o \, f_A)(x) = P_n(f_A(x)) = P_n(A(x)) = A_n(x)$ for all $n \in \mathbb{N}_0$ by Remark 2.2, the linear map f_A is continuous by Corollary 2.1. □

Definition 2.2. *Let $X \supset \phi$ be an FK space. Then X is said to have*

(a) *AD if ϕ is dense in X,*

(b) *AK if every sequence $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is, if every sequence x is the limit of its m-sections*

$$x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)}.$$

If an FK space has AK or AD we also say that it is an AK or AD space.

Remark 2.3. *[16, Remark 1.19.] Every AK space has AD. The converse is not true in general.*

Example 2.2. *The spaces ω, c_0 and ℓ_p $(1 \leq p < \infty)$ all have AK by Example 1.1 and Theorem 1.4.*

The FK metric of an FK space will turn out to be unique.

Theorem 2.3. *Let X and Y be FK spaces and $X \subset Y$. Then the metric d_X on X is stronger than the metric $d_Y|_X$ of Y on X. The metrics are equivalent if and only if X is a closed subspace of Y. In particular, the metric of an FK space is unique, this means there is at most one way to make a linear subspace of ω into an FK space.*

Proof. Let $\iota : (X, d_X) \to (Y, d_Y)$ be the inclusion map. Since X is an FK space, $\iota : (X, d_X) \to (Y, d|_Y)$ is continuous, and so is $\iota : (X, d_X) \to (Y, d_Y)$ by Theorem 2.1. Thus d_X is stronger than $d_Y|_X$. The uniqueness of an FK space is shown in exactly the same way. Let X be closed in Y, then X becomes an FK space with $d_Y|_X$, and the uniqueness of an FK metric implies that d_X and $d_Y|_X$ are equivalent. Conversely, if d_X and $d_Y|_X$ are equivalent, then X is a complete subspace of Y , hence a closed subspace of Y . □

Example 2.3. *The BK spaces c_0 and c are closed subspaces of ℓ_∞. Thus the BK norms on c_0, c and ℓ_∞ must be the same. The BK space ℓ_1 is a subspace of ℓ_∞ which is not closed in ℓ_∞. Thus its BK norm $||.||_1$ is strictly stronger than the BK norm $||.||_\infty$ on ℓ_∞.*

CHAPTER 3

THE GENERAL THEORY OF MEASURES OF NONCOMPACTNESS

In this chapter, we give a comprehensive survey of measures of noncompactness starting with an axiomatic approach as in [17]. Then in the following sections, we consider two different measures of noncompactness : Kuratowski and Hausdorff measures of noncompactness.

3.1 AN AXIOMATIC APPROACH TO A MEASURE OF NONCOMPACTNESS

In this section, we list the axioms for the general notion of measures of noncompactness. We follow the general idea that the best way of studying measures of noncompactness is to take an axiomatic approach. The two requirements for the sets of axioms are that they should have natural realizations and be useful tools for applications.

Definition 3.1. *[17, Definition 1.1] Let (X, d) be a complete metric space and $\mathcal{M}_X$ be the family of all non-empty bounded subsets of X. A map*

$$\phi : \mathcal{M}_X \to [0, +\infty)$$

is called a measure of noncompactness *defined on X if it satisfies the following properties:*

(i) Regularity *:* $\phi(Q) = 0$ *if and only if* Q *is a relatively compact set.*

(ii) Invariant under closure *:* $\phi(Q) = \phi(\overline{Q})$, *for all* $Q \in \mathcal{M}_X$.

(iii) Semi-additivity *:* $\phi(Q_1 \cup Q_2) = \max\{\phi(Q_1), \phi(Q_2)\}$ *, for all* $Q_1, Q_2 \in \mathcal{M}_X$.

From these axioms, the following properties can be deduced directly:

Theorem 3.1. **(1)** Monotonicity *:* $Q_1 \subset Q_2$ *implies* $\phi(Q_1) \leq \phi(Q_2)$.

(2) $\phi(Q_1 \cap Q_2) \leq \min\{\phi(Q_1), \phi(Q_2)\}$ *, for all* $Q_1, Q_2 \in \mathcal{M}_X$.

(3) Non - singularity *: If* Q *is a finite set then* $\phi(Q) = 0$.

(4) *Generalized Cantor's Intersection Theorem: If* (Q_n) *is a decreasing sequence of nonempty, closed and bounded subsets of* X *and* $\lim_{n\to} \phi(Q_n) = 0$, *then the intersection* Q_∞ *of all* Q_n *is nonempty and compact.*

Moreover, if X *is a Banach space, a measure of noncompactness* ϕ *can be satisfy some additional properties. We mention some of them:*

(5) Semi-homogeneity *:* $\phi(tQ) = |t|\phi(Q)$ *for all scalars* t *and all* $Q \in \mathcal{M}_X$.

(6) Algebraic semi-additivity *:* $\phi(Q_1 + Q_2) \leq \phi(Q_1) + \phi(Q_2)$, *for all* $Q_1, Q_2 \in \mathcal{M}_X$.

(7) Invariance under translations *:* $\phi(x + Q) = \phi(Q)$ *for any* $x \in X$ *and* $Q \in \mathcal{M}_X$.

(8) Invariance under passage to the convex hull *:* $\phi(Q) = \phi(co(Q))$, *for all* $Q \in \mathcal{M}_X$.

Example 3.1. *[17, Example 1.] In every metric space X, the map*

$$\phi_1(Q) = \begin{cases} 0, & \textit{if } Q \textit{ is relatively compact;} \\ 1, & \textit{otherwise.} \end{cases}$$

is a measure of noncompactness. Furthermore ϕ_1 *is called the* discrete measure *of noncompactness. This measure is algebraically semi-additive and invariant under translations and passage to the convex hull in normed spaces.*

3.2 TWO MEASURES OF NONCOMPACTNESS

In this section we define the two most frequently used measures of noncompactness: the Kuratowski measure of noncompactness and the Hausdorff measure of noncompactness. We state their properties and give the relations between them.

3.2.1 The Kuratowski measure of noncompactness and its properties

The notion of measure of noncompactness was first mentioned by Kuratowski [20] in 1930. He defined this new measure in connection with general topological problems.

Definition 3.2. *Let (X, d) be a metric space and $\mathcal{M}_X$ be the family of all nonempty bounded subsets of X. The Kuratowski measure of noncompactness of $Q \in \mathcal{M}_X$, denoted by $\alpha(Q)$, is the infimum of all positive ε such that Q can be covered by finitely many sets of diameters less than ε; that is, the function $\alpha : \mathcal{M}_X \to [0, \infty)$ is defined as follows*

$$\alpha(Q) = inf\left\{\varepsilon > 0 : Q \subset \bigcup_{k=1}^{n} S_k, S_k \subset X, diam(S_k) < \varepsilon(k = 1, ..., n; n \in N)\right\}$$

We start with some results from Kuratowski:

Lemma 3.1. *Let (X, d) be a complete metric space, and $Q, Q_1, Q_2 \subset \mathcal{M}_X$. Then we have*

$$\alpha(Q) = 0 \text{ if and only if } Q \text{ is relatively compact }, \tag{3.1}$$

$$Q_1 \subset Q_2 \text{ implies } \alpha(Q_1) \leq \alpha(Q_2), \tag{3.2}$$

$$\alpha(Q) = \alpha(\overline{Q}), \tag{3.3}$$

$$\alpha(Q_1 \cup Q_2) = \max\{\alpha(Q_1), \alpha(Q_2)\}, \tag{3.4}$$

$$\alpha(Q_1 \cap Q2) \leq \min\{\alpha(Q_1), \alpha(Q_2)\}. \tag{3.5}$$

Proof. In view of the axiomatic approach we mentioned before, the properties (3.1), (3.3) and (3.4) guarantee that the function α is indeed a measure of noncompactness.

(i) We prove (3.1). By Definition 3.2, $\alpha(Q) = 0$ directly implies $\overline{Q}$ is compact. We assume that Q is relatively compact. Since X is a complete metric space, Q is totally bounded. Then, for each $\varepsilon > 0$, Q can be covered by finitely many sets with diameter smaller than or equal to ε. Since $\varepsilon > 0$ is arbitrary, $\alpha(Q) = 0$.

(ii) The statement in (3.2) comes from Definition 3.2 as follows:

We assume $\alpha(Q_2) = m$ and $Q_1 \subset Q_2$. Let $\varepsilon > 0$ be given. Then these exist subsets S_k of which $diam(S_k) \leq m + \varepsilon$ $(k = 1, 2, \ldots, n)$ such that $Q_2 \subset \bigcup_{k=1}^{n} S_k$. Now $Q_1 \subset Q_2$ implies $Q_1 \subset \bigcup_{k=1}^{n} S_k$ and so $\alpha(Q_1) \leq m + \varepsilon$. Since $\varepsilon > 0$ is arbitrary $\alpha(Q_1) \leq m$.

(iii) Now we show (3.3). We have $\alpha(Q) \leq \alpha(\overline{Q})$ by (3.2). To prove converse inequality, let $\alpha(Q) = m$ and $\varepsilon > 0$ be given. Then there exist subsets S_k with $diam(S_k) \leq m + \varepsilon$ $(k = 1, 2, \ldots, n)$ such that

$$Q \subset \bigcup_{k=1}^{n} S_k.$$

This implies

$$\overline{Q} \subset \overline{\bigcup_{k=1}^{n} S_k} = \bigcup_{k=1}^{n} \overline{S_k}.$$

and since $diam(S_k) = diam(\overline{S_k})$ $(k = 1, 2, \ldots, n)$, we conclude $\alpha(\overline{Q}) \leq m = \alpha(Q)$. Hence the equality in (3.3) holds.

(iv) It follows from (3.2) that,

$$\alpha(Q_1) \leq \alpha(Q_1 \cup Q_2) \text{ and } \alpha(Q_2) \leq \alpha(Q1 \cup Q_2),$$

and so

$$\max\{\alpha(Q_1), \alpha(Q_2)\} \leq \alpha(Q_1 \cup Q_2). \tag{3.6}$$

Let $\max\{\alpha(Q_1), \alpha(Q_2)\} = m$, and $\varepsilon > 0$ be given. We know that Q_1 and Q_2 can be covered by a finite number of subsets of diameter smaller than $m + \varepsilon$ (Definition 3.2). Obviously, the union of these covers is a finite cover of $Q_1 \cup Q_2$. Hence we have $\alpha(Q_1 \cup Q_2) \leq m + \varepsilon$. Since ε is arbitrary, we obtain

$$\alpha(Q_1 \cup Q_2) \leq \max\{\alpha(Q_1), \alpha(Q_1)\}. \tag{3.7}$$

Now the equality in (3.4) follows from (3.6) and (3.7)

(v) Finally, we show (3.5).

From

$$Q_1 \cap Q_2 \subset Q_1 \text{ and } Q_1 \cap Q_2 \subset Q_2,$$

we obtain by (3.2)

$$\alpha(Q_1 \cap Q_2) \leq \alpha(Q_1) \text{ and } \alpha(Q_1 \cap Q_2) \leq \alpha(Q_2).$$

Hence we have the inequality

$$\alpha(Q_1 \cap Q_2) \leq \min \{\alpha(Q_1), \alpha(Q_2)\}.$$

□

The next theorem is a generalization of the well-known Cantor Intersection Theorem.

Theorem 3.2. *(Kuratowski). Let (X, d) be a complete metric space. If $\{Q_n\}$ is a decreasing sequence of nonempty, closed and bounded subsets of X and $\lim_{n\to\infty} \alpha(Q_n) = 0$, then the intersection Q_∞ of all Q_n is a nonempty and compact subset of X.*

Proof. First, we show $Q_\infty \neq \emptyset$. Let $x_n \in Q_n$ and $X_n = \{x_k : k \geq n\}$ for $n = 1, 2, \ldots$. Since $X_n \subset Q_n$, we obtain from (3.1), (3.2) and (3.4)

$$\alpha(X_1) = \alpha(X_n) \leq \alpha(Q_n) \text{ for each } n. \tag{3.8}$$

The assumption of the theorem and (3.8) together imply $\alpha(X_1) = 0$, hence X_1 is a relatively compact set, that is, $\overline{X_1}$ is compact. Thus the sequence $(x_n)_{n=1}^{\infty}$ has a convergent subsequence with limit $x \in X$, say. Since Q_n is closed in X, we get $x \in Q_n$ for all $n = 1, 2, \ldots$, that is, $x \in Q_\infty$.

The set Q_∞ is a closed set being the intersection of closed sets Q_n of X. Since $Q_\infty \subset Q_n$ for all $n = 1, 2, \ldots$, we obtain from (3.2) that $\alpha(Q_\infty) \leq \alpha(Q_n)$ $(\star)$. Now $(\star)$ and $\lim_{n\to\infty} \alpha(Q_n) = 0$ together imply $\alpha(Q_\infty) = 0$, hence Q_∞ is a relatively compact set by (3.1), that is, $\overline{Q}_\infty$ is a compact set. Since Q_∞ is closed, $Q_\infty = \overline{Q}_\infty$. Therefore, Q_∞ is compact. □

Theorem 3.3. *Let X be a normed space and $Q, Q_1, Q_2 \subset \mathcal{M}_X$. Then we have*

$$\alpha(Q_1 + Q_2) \leq \alpha(Q_1) + \alpha(Q_2), \tag{3.9}$$

$$\alpha(Q + x) = \alpha(Q) \text{ for each } x \in X, \tag{3.10}$$

$$\alpha(\lambda Q) = |\lambda|\alpha(Q) \text{ for each } \lambda \text{ scalar }, \tag{3.11}$$

$$\alpha(Q) = \alpha(co(Q)). \tag{3.12}$$

Proof. (i) We prove (3.9). Let $S_k \in \mathcal{M}_X$ with $diam(S_k) < d$ for each $k = 1, \dots, n$ and $Q_1 \subset \bigcup_{k=1}^{n} S_k$. Similarly, let $T_j \subset \mathcal{M}_X$ with $diam(T_j) < s$ for each $j = 1, \dots, m$ and $Q_2 \subset \bigcup_{j=1}^{m} T_j$. Then we have

$$Q_1 + Q_2 \subset \bigcup_{k=1}^{n} \bigcup_{j=1}^{m} (S_k + T_j) \text{ and } diam(S_k + T_j) < d + s. \tag{3.13}$$

It follows from (3.13) that $\alpha(Q_1 + Q_2) < d + s$.

(ii) Now we prove (3.10). Let $x \in X$ be given. It follows from (3.9) that

$$\alpha(Q + x) \leq \alpha(Q) + \alpha(\{x\}) = \alpha(Q), \tag{3.14}$$

and by the same argument we have

$$\alpha(Q) = \alpha((Q + x) + (-x)) \leq \alpha(Q + x) + \alpha(\{-x\}) = \alpha(Q + x). \tag{3.15}$$

Now we obtain (3.10) from (3.14) and (3.15).

(iii) The equality in (3.11) is obvious for $\lambda = 0$. So let $\lambda \neq 0$ and $S_k \in \mathcal{M}_X$ with $diam(S_k) < d$ for $k = 1, \dots, n$ and $Q_1 \subset \bigcup_{k=1}^{n} S_k$. Then for any scalar λ, $\lambda Q \subset \bigcup_{k=1}^{n} \lambda S_k$ and $diam(\lambda S_k) = |\lambda| diam(S_k)$. Hence it follows that $\alpha(\lambda Q) \leq |\lambda|\alpha(Q)$. Since $\lambda \neq 0$, analogously we have $\alpha(Q) = \alpha(\lambda^{-1}(\lambda Q)) \leq |\lambda^{-1}|\alpha(\lambda(Q))$, that is $|\lambda|\alpha(Q) \leq \alpha(\lambda Q)$. This proves (3.11).

(iv) Since $Q \subset co(Q), \alpha(Q) \leq \alpha(co(Q))$ is clear from (3.2). For the converse inequality, let $S_k \in \mathcal{M}_X$ with $diam(S_k) < d$ for each $k = 1, \dots, n$ and $Q = \bigcup_{k=1}^{n} S_k$. It follows by (1.10) that

$$co(Q) = \left\{ \sum_{k=1}^{n} \lambda_k x_k : \lambda_k \geq 0 \; (k = 1, \dots, n), \sum_{k=1}^{n} \lambda_k = 1, x_k \in co(S_k) \right\}. \tag{3.16}$$

Let $\varepsilon > 0$ and

$$S = \left\{ (\lambda_1, ..., \lambda_n) : \sum_{k=1}^{n} \lambda_k = 1,\ \lambda_k \geq 0,\ k = 1, \ldots, n \right\}$$

Since S is a closed and bounded subset of $(R^n, ||.||_\infty)$, it is compact by the Heine-Borel Theorem (Theorem 8.7 in Appendix), and the norm is defined as follows

$$||(\lambda_1, ..., \lambda_n)||_\infty = \sup_{1 \leq i \leq n} |\lambda_i|.$$

We put $M = \sup\{||x|| : x \in \sum_{k=1}^{n} co(S_k)\}$. Let

$$T = \{(t_{j,1}, ..., t_{j,n}) : j = 1, \ldots, m\} \subset S$$

be a finite $\varepsilon/(Mn)$-net for S, with respect to the $||.||_\infty$-norm. Hence, if $\sum_{k=1}^{n} \lambda_k x_k$ is a convex combination of elements of Q, where $x_k \subset co(S_k)$ for $k = 1, \ldots, n$, then there exist $(t_{j,1}, ..., t_{j,n}) \subset T$ such that

$$||(\lambda_1, ..., \lambda_n) - (t_{j,1}, ..., t_{j,n})||_\infty < \frac{\varepsilon}{Mn}. \tag{3.17}$$

Since

$$\sum_{k=1}^{n} \lambda_k x_k = \sum_{k=1}^{n} t_{j,k} x_k + \sum_{k=1}^{n} (\lambda_k - t_{j,k}) x_k \tag{3.18}$$

it follows from (3.16), (3.17) and (3.18) that

$$\begin{aligned}
co(Q) &= \left\{ \sum_{k=1}^{n} \lambda_k x_k : \lambda_k \geq 0\ (k = 1, \ldots, n),\ \sum_{k=1}^{n} \lambda_k = 1,\ x_k \in co(S_k) \right\} \\
&\subset \bigcup_{j=1}^{m} \left\{ \sum_{k=1}^{n} (\lambda_k - t_{j,k} + t_{j,k}) x_k : \lambda_k \geq 0\ (k = 1, \ldots, n),\ \sum_{k=1}^{n} \lambda_k = 1,\ x_k \in co(S_k) \right\} \\
&= \bigcup_{j=1}^{m} \left\{ \sum_{k=1}^{n} t_{j,k} x_k \right\} + \bigcup_{j=1}^{m} \left\{ \sum_{k=1}^{n} (\lambda_k - t_{j,k}) x_k \right\} \\
&\subset \bigcup_{j=1}^{m} \left\{ \sum_{k=1}^{n} t_{j,k} co(S_k) \right\} + \bigcup_{j=1}^{m} \left\{ \sum_{k=1}^{n} \sup_j |\lambda_k - t_{j,k}| x_k \right\} \\
&\subset \bigcup_{j=1}^{m} \left\{ \sum_{k=1}^{n} t_{j,k} co(S_k) \right\} + \frac{\varepsilon}{Mn} \left\{ \sum_{k=1}^{n} x_k : x_k \in co(S_k) \right\}.
\end{aligned}$$

Therefore

$$co(Q) \subset \bigcup_{j=1}^{m} \left\{ \sum_{k=1}^{n} t_{j,k} co(S_k) \right\} + \frac{\varepsilon}{Mn} \sum_{k=1}^{n} B_k \tag{3.19}$$

where $B_k = \{x \in X : ||x|| \leq M\}$ for $k = 1, ..., n$. Now from the previous results (1.10), (3.9), (3.19) we have

$$\begin{aligned}
\alpha(co(Q)) &\leq \alpha\left(\bigcup_{j=1}^{m}\left\{\sum_{k=1}^{n} t_{j,k}co(S_k)\right\}\right) + \alpha\left(\frac{\varepsilon}{Mn}\sum_{k=1}^{n} B_k\right) \\
&\leq \max_{1\leq j\leq m} \alpha\left(\sum_{k=1}^{n} t_{j,k}co(S_k)\right) + \frac{\varepsilon}{Mn}\sum_{k=1}^{n} 2\alpha\left(B_k\right) \\
&\leq \max_{1\leq j\leq m} \sum_{k=1}^{n} t_{j,k}\alpha\left(co(S_k)\right) + \frac{\varepsilon}{Mn} 2nM \\
&< d \max_{1\leq j\leq m} \sum_{k=1}^{n} t_{j,k} + 2\varepsilon < d + 2\varepsilon.
\end{aligned}$$

□

3.2.2 The Hausdorff measure of noncompactness and its properties

Here we study the Hausdorff measure of noncompactness. Its basic properties are analogous to those of the Kuratowski measure of noncompactness stated in Lemma 3.1 and Theorem 3.3 We start with the definition and then state the properties.

Definition 3.3. *The* Hausdorff or ball measure of noncompactness *of a bounded set Q in a metric space X, denoted by $\chi(Q)$, is the infimum of all positive ε such that Q can be covered by finitely many open balls of radius less than ε, that is, the function $\chi : \mathcal{M}_X \to [0, \infty)$ is defined by*

$$\chi(Q) = \inf\left\{\varepsilon > 0 : Q \subset \bigcup_{k=1}^{n} B_{r_k}(x_k),\ x_k \in X,\ r_k < \varepsilon\,(k = 1, \ldots, n; n \in N)\right\}.$$

The next results are analogous to the respective one for the Kuratowski measure of noncompactness, and their proof are similar.

Lemma 3.2. *Let (X, d) be a complete metric space, and $Q, Q_1, Q_2 \subset \mathcal{M}_X$. Then* *we have*

$$\chi(Q) = 0 \text{ if and only if } Q \text{ is relatively compact,}$$
$$Q_1 \subset Q_2 \text{ implies } \chi(Q_1) \leq \chi(Q_2),$$
$$\chi(Q) = \chi(\overline{Q}),$$
$$\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\},$$
$$\chi(Q_1 \cap Q2) \leq \min\{\chi(Q_1), \chi(Q_2)\}.$$

Theorem 3.4. *Let X be a normed space and $Q, Q_1, Q_2 \subset \mathcal{M}_X$. Then we have*

$$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2),$$
$$\chi(Q + x) = \chi(Q) \text{ for each } x \in X,$$
$$\chi(\lambda Q) = |\lambda|\chi(Q) \text{ for each } \lambda \text{ scalar },$$
$$\chi(Q) = \chi(co(Q)).$$

Now we state a theorem from *[17]:*

Theorem 3.5. *[17, Theorem 2.5] Let $B_1(0)$ be the unit ball in an infinite dimensional Banach space X. Then $\chi(B_1(0)) = 1$.*

Proof. The inequality $\chi(B_1(0)) \leq 1$ is clear from the definition of the Hausdorff measure of noncompactness (Definition 3.3). We assume that $d = \chi(B_1(0)) < 1$. Let $\varepsilon > 0$ be chosen such that $d + \varepsilon < 1$. Then there exist $x_1, x_2, \ldots, x_m$ in X such that

$$B_1(0) \subset \bigcup_{k=1}^{m} B(x_k, (d+\varepsilon)) = \bigcup_{k=1}^{m} (x_k + (d+\varepsilon)B_1(0)).$$

It follows from the properties of Hausdorff measure of noncompactness that

$$d = \chi(B_1(0)) \leq \max_{1\leq k\leq m} \{\chi(x_k), (d+\varepsilon)\chi(B_1(0))\} = (d+\varepsilon)\chi(B_1(0)) = d(d+\varepsilon).$$

This implies $d = \chi(B_1(0)) = 0$ and hence $B_1(0)$ is relatively compact, that is, it is totally bounded, which contradicts the infinite dimensionality of the space X. Therefore $\chi(B_1(0)) = 1$. $\square$

Now we shall show how to compute the Hausdorff measure of noncompactness in the spaces ℓ_p for $1 \leq p < \infty$ and c_0.

Theorem 3.6. *(The Hausdorff Measure of Noncompactness in the spaces ℓ_p and c_0.) Let Q be a bounded subset of the normed space X where $X = \ell_p$ for $1 \leq p < \infty$ or $X = c_0$. If $P_n : X \to X$ is the operator defined by*

$$P_n(x) = \sum_{k=1}^{n} x_k e^{(k)} = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \text{ for } x = (x_k)_{k=1}^{\infty} \in X,$$

then we have

$$\chi(Q) = \lim_{n\to\infty} \left(\sup_{x\in Q} ||(I - P_n)(x)|| \right). \tag{3.20}$$

Proof. We clearly have $\forall n \in \mathbb{N}$

$$Q \subset P_n Q + (I - P_n)Q. \tag{3.21}$$

It follows from Lemma 3.2 and Theorem 3.4 that

$$\begin{aligned}
\chi(Q) &\leq \chi(P_n(Q) + (I - P_n)(Q)) \\
&\leq \chi(P_n(Q)) + \chi((I - P_n)(Q)) \\
&= \chi((I - P_n)(Q)) \\
&\leq \sup_{x\in Q} ||(I - P_n)(x)||.
\end{aligned}$$

Therefore

$$\chi(Q) \leq \sup_{x\in Q} ||(I - P_n)(x)||. \tag{3.22}$$

Since the limit in (3.20) clearly exists, we have by (3.22)

$$\chi(Q) \leq \lim_{n\to\infty} \left(\sup_{x\in Q} ||(I - P_n)(x)|| \right). \tag{3.23}$$

We prove the converse inequality in (3.23). Let $\varepsilon > 0$ and $\{z_1, \ldots, z_k\}$ be a $(\chi(Q) + \varepsilon)$-net of Q. By (1.19)

$$Q \subset z_1, \ldots, z_k + (\chi(Q) + \varepsilon)B_1(0). \tag{3.24}$$

It follows from (3.24) that for any $x \subset Q$ there exist $z \in \{z_1, ..., z_k\}$ and $s \in B_1(0)$ such that $x = z + (\chi(Q) + \varepsilon)s$. Hence we have

$$\sup_{x\in Q} ||(I - P_n)(x)|| \leq \sup_{1\leq j\leq k} ||(I - P_n)(z_j)|| + (\chi(Q) + \varepsilon). \tag{3.25}$$

Finally, (3.25) implies

$$\lim_{n\to\infty} \left(\sup_{x\in Q} ||(I - P_n)(x)|| \right) \leq \chi(Q) + \varepsilon.$$

and, since $\varepsilon > 0$ was arbitrary, we have

$$\lim_{n\to\infty}\left(\sup_{x\in Q}||(I-P_n)(x)||\right) \leq \chi(Q).$$

This and (3.22) imply (3.20). $\square$

Theorem 3.7. *[16, Theorem 2.23] Let X be a Banach space with a Schauder basis $\{e_1, e_2, \ldots\}$, Q be a bounded subset of X, and $P_n : X \mapsto X$ the projector onto the linear span of $\{e_1, e_2, \ldots, e_n\}$. Then*

$$\frac{1}{a}\limsup_{n\to\infty}\left(\sup_{x\in Q}||(I-P_n)(x)||\right) \leq \chi(Q) \tag{3.26}$$

$$\leq \inf_n \sup_{x\in Q}||(I-P_n)(x)|| \leq \limsup_{n\to\infty}\left(\sup_{x\in Q}||(I-P_n)(x)||\right)$$

where $a = \limsup_{n\to\infty}\|I-P_n\|$.

3.2.3 Relations between the Kuratowski and the Hausdorff measures of noncompactness

The next result shows that the functions α and χ are somehow equivalent.

Theorem 3.8. *[16, Theorem 2.13] Let (X, d) be a metric space and $Q \in \mathcal{M}_X$. Then we have*

$$\chi(Q) \leq \alpha(Q) \leq 2\chi(Q). \tag{3.27}$$

Proof. Let $\varepsilon > 0$ be given.
If $\{x_1, ..., x_n\}$ is an ε-net of Q, then $\{Q \cap B_\varepsilon(x_k) : k = 1, ..., n\}$ is a cover of Q with sets of diameter less than 2ε. This shows $\alpha(Q) \leq 2\chi(Q)$. To prove the inequality on the left hand side of (3.27), we assume that $\{S_k : k = 1, \ldots, n\}$ is a cover of Q with sets of diameter less than ε and consider $y_k \in S_k$ for $k = 1, ..., n$. Now $\{y_1, ..., y_n\}$ is an ε-net of Q. This proves $\chi(Q) \leq \alpha(Q)$. $\square$

Remark 3.1. *In general, the inequalities in (3.27) can be shown to be best possible. The geometric properties of the space are directly related to the two measures of noncompactness and it is possible to improve the inequality $\chi(Q) \leq \alpha(Q)$ in certain spaces. For example, in Hilbert space,*

$$\sqrt{2}\chi(Q) \leq \alpha(Q) \leq 2\chi(Q),$$

and in ℓ_p *for* $1 \leq p < \infty$,

$$\sqrt[p]{2}\chi(Q) \leq \alpha(Q) \leq 2\chi(Q).$$

Remark 3.2. *In general,* α *and* χ *are different measures of noncompactness. However, we can find a direct relation between them in some Banach spaces.*

CHAPTER 4

THE CLASSICAL SEQUENCE SPACES AND CHARACTERIZATIONS OF MATRIX TRANSFORMATIONS

Now we establish a relationship between the β- and continuous duals of an FK space. We use the follows notations

$$X^{\sharp} = \{f|f : X \longrightarrow \mathbb{C}, \textit{linear}\}$$
$$X^{'} = \{f|f : X \longrightarrow \mathbb{C}, \textit{ continuous } \}$$
$$X^{f} = \left\{(f(e^{k}))|f \in X^{'}\right\}$$

If $X \subset \omega$ is a linear metric space and $a \in \omega$, then we write

$$||a||^{*}_{\delta} = ||a||^{*}_{X,\delta} = \sup_{x\in \overline{B}_{\delta}(0)} \left|\sum_{k=0}^{\infty} a_k x_k\right| ;$$

provided the expression on the right hand exists and is finite which is the case whenever X is an FK space and the series $\sum_{k=0}^{\infty} a_k x_k$ converge for all $x \in X$ *(Remark 2.2)*.

Theorem 4.1. *[16, Theorem 1.23(b)] Let X be an FK space. Then we have $A \in (X, \ell_{\infty})$ if and only if*

$$||A||^{*}_{\delta} = \sup_{n} ||A_n||^{*}_{\delta} < \infty \textit{ for some } \delta > 0$$

where $A_n = (a_{nk})_{k=0}^{\infty}$ denotes the sequence in the n-th row of the matrix A.

Theorem 4.2. *Let X and Y be BK spaces.*

(a) *[16, Theorem 1.23(a)] Then $(X,Y) \subset B(X,Y)$, that is, every $A \in (X,Y)$ defines an operator $L_A \in B(X,Y)$ by $L_A(x) = Ax$ for all $x \in X$.*

(b) *[24, Theorem 1.9] If X has AK then $B(X,Y) \subset (X,Y)$, that is, for each $L \in B(X,Y)$ there exists $A \in (X,Y)$ such that $L_A(x) = Ax$ for all $x \in X$ holds.*

(c) *[16, Theorem 1.23(b)] We have $A \in (X, \ell_\infty)$ if and only if*

$$||A||_{(X,\ell_\infty)} = \sup_n(\sup\{|A_n x| : ||x|| = 1\}) < \infty;$$

if $A \in (X, \ell_\infty)$ then

$$||L_A|| = ||A||_{(X,\ell_\infty)}.$$

Proof. **(a)** This is Theorem 2.2

(b) Let $L \in B(X;Y)$ be given. We write $L_n = P_n o L$ for all n, and put $a_{nk} = L_n(e^{(k)})$ for all n and k. Let $x = (x_k)_{k=0}^\infty \in X$ be given. Since X has AK, we have $x = \sum_{k=0}^\infty x_k e^{(k)}$, and since Y is a BK space, it follows that $L_n \in X'$ for all n. Hence we obtain $L_n(x) = \sum_{k=0}^\infty x_k L_n(e^{(k)}) = \sum_{k=0}^\infty x_k a_{nk} = A_n x$ for all n, and so $L(x) = Ax$.

(c) This follows immediately from Theorem 4.1 and the definition of $||A||_{(X,\ell_\infty)}$.

□

Theorem 4.3. *[23, 8.3.6] Let X be an FK space with AD, and Y and Y_1 be FK spaces with Y_1 a closed subspace of Y. Then $A \in (X, Y_1)$ if and only if $A \in (X,Y)$ and $Ae^{(k)} \in Y_1$ for all k.*

Theorem 4.4. *[23, Theorem 7.2.9] Let $X \supset \phi$ be an FK space. Then $X^\beta \subset X'$; this means that there is a linear one-to-one map $T : X^\beta \to X'$. If X has AK then T is onto.*

Proof. We define the map T by $Ta = f_a$ $(a \in X^\beta)$ where f_a is the functional with $f_a(x) = \sum_{k=0}^\infty a_k x_k$ for all $x \in X$, and observe that $Ta = f_a \in X'$ for all

$a \in X^\beta$ by Remark 2.2. Obviously T is linear. Furthermore, if $Ta = 0$ then $f_a(x) = \sum_{k=0}^{\infty} a_k x_k = 0$ for all $x \in X$, in particular $f_a(e^{(k)}) = a_k = 0$ for all k, that is, $a = 0$. Thus $Ta = 0$ implies $a = 0$, and consequently T is one-to-one. Now we assume that X has AK. Let $f \in X'$ be given. We define the sequence a by $a_k = f(e^{(k)})$ for $k = 0, 1, \dots$. Let $x \in X$ be given. Then $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, since X has AK, and $f \in X'$ implies $f(x) = f(\sum_{k=0}^{\infty} x_k e^{(k)}) = \sum_{k=0}^{\infty} x_k f(e^{(k)}) = \sum_{k=0}^{\infty} a_k x_k$, hence $a \in X^\beta$ and $Ta = f$. This shows that the map T is onto. □

A relation between the functional and continuous duals of an FK space is given by

Theorem 4.5. *Let $X \supset \phi$ be an FK space.*

(a) *Then the map $q : X' \to X^f$ given by $q(f) = (f(e^{(k)}))_{k=0}^{\infty}$ is onto. Moreover, if $T : X^\beta \to X'$ denotes the map of Theorem 4.4, then $q(Ta) = a$ for all $a \in X^\beta$ [23, Theorem 7.2.10].*

(b) *Then $X^f = X'$, that is, the map q of Part (a) is one-to-one, if and only if X has AD [23, Theorem 7.2.12].*

Proof. **(a)** Let $a \in X^f$ be given. Then there is $f \in X'$ such that $a_k = f(e^{(k)})$ for all k, and so $q(f) = (f(e^{(k)}))_{k=0}^{\infty} = a$. This shows that q is onto. Now let $a \in X^\beta$ be given. We put $f = Ta \in X'$ and obtain $q(Ta) = q(f) = (f(e^{(k)}))_{k=0}^{\infty} = ((Ta)(e^{(k)}))_{k=0}^{\infty} = (a_k)_{k=0}^{\infty} = a$.

(b) First we assume that X has AD. Then $q(f) = 0$ implies $f = 0$ on ϕ, hence $f = 0$, since X has AD. This shows that q is one-to-one. Conversely we assume that X does not have AD. By the Hahn-Banach theorem (Theorem 8.8 in Appendix), there exists an $f \in X'$ with $f \neq 0$ and $f = 0$ on ϕ. Then we have $q(f) = 0$, and q is not one-to-one.

□

Theorem 4.6. *[23, Theorem 4.3.15] Let $X \supset \phi$ and $Y \supset \phi$ be BK spaces. Then $Z = M(X, Y)$ is a BK space with*

$$||z|| = \sup_{x \in S_X} ||xz|| \text{ for all } z \in Z.$$

We obtain as an immediate consequence of Theorem *4.6*

Corollary 4.1. *[23, 4.3.16] The $\alpha-$, $\beta-$ and $\gamma-$ duals of a BK space X are BK spaces with*

$$||a||_\alpha = \sup_{x\in S_X} ||ax||_1 = \sup_{x\in S_X} \left(\sum_{k=0}^{\infty} |a_k x_k|\right) \text{ for all } x \in X^\alpha$$

and

$$||a||_\beta = \sup_{x\in S_X} ||ax||_{bs} = \sup_{x\in S_X} \left(\sup_n \left|\sum_{k=0}^{n} a_k x_k\right|\right) \text{ for all } x \in X^\beta, X^\gamma.$$

Furthermore, X^β is a closed subspace of X^γ.

Also, let X be any of the spaces c_0, c, ℓ_∞ or ℓ_p $(1 \le p < \infty)$. Then, we have $||.||^*_X = ||.||_{X^\beta}$ on X^β , where $||.||_{X^\beta}$ denotes the natural norm on the dual space X^β .

Lemma 4.1. *Let $\ddagger$ denote any of the symbols α, β or γ . Then, we have $c_0^\ddagger = c^\ddagger = \ell_\infty^\ddagger = \ell_1$, $\ell_1^\ddagger = \ell_\infty$ and $\ell_p^\ddagger = \ell_q$, where $1 < p < \infty$ and $q = p/(p-1)$.*

Now, we give the following theorem without proof. You can find the proof of the following theorem in [23]

Theorem 4.7. *[10, 3.Ergebnisse] Let $1 < p, r < \infty$, $q = p/(p-1)$ and $s = r/(r-1)$. Then the necessary and suficient conditions for $A \in (X, Y)$ can be read from the following table:*

From/To	ℓ_∞	c_0	c	ℓ_1	ℓ_r
ℓ_∞	*1*	*4*	*9*	*14*	*17*
c_0	*1*	*5*	*10*	*14*	*17*
c	*1*	*6*	*11*	*14*	*17*
ℓ_1	*2*	*7*	*12*	*15*	*18*
ℓ_p	*3*	*8*	*13*	*16*	*unknown*

where

1 $A \in (\ell_\infty, \ell_\infty) = (c, \ell_\infty) = (c_0, \ell_\infty) \Leftrightarrow (1.1)$.

$$\textit{(1.1)} \quad \sup_n \sum_k |a_{nk}| < \infty$$

2 $A \in (\ell_1, \ell_\infty) \Leftrightarrow (2.1)$.

$$\textit{(2.1)} \quad \sup_{n,k} |a_{nk}| < \infty.$$

3 $A \in (\ell_p, \ell_\infty) \Leftrightarrow (3.1)$.

$$\textit{(3.1)} \quad \sup_n \sum_k |a_{nk}|^q < \infty.$$

4 $A \in (\ell_\infty, c_0) \Leftrightarrow (4.1)$.

$$\textit{(4.1)} \quad \lim_n \sum_k |a_{nk}| = 0.$$

5 $A \in (c_0, c_0) \Leftrightarrow (1.1), (5.1)$.

$$\textit{(5.1)} \quad \lim_n a_{nk} = 0 \textit{ for all } k.$$

6 $A \in (c, c_0) \Leftrightarrow (1.1), (5.1), (6.1)$.

$$\textit{(6.1)} \quad \lim_n \sum_k a_{nk} = 0.$$

7 $A \in (\ell_1, c_0) \Leftrightarrow (2.1), (5.1)$.

8 $A \in (\ell_p, c_0) \Leftrightarrow (3.1), (5.1)$.

9 $A \in (\ell_\infty, c) \Leftrightarrow (9.1), (9.2) \Leftrightarrow (1.1), (9.1), (9.3) \Leftrightarrow (9.1), (9.4)$.

$$\textit{(9.1)} \quad \lim_n a_{nk} \textit{ exists for all } k,$$

$$\textit{(9.2)} \quad \lim_n \sum_k |a_{nk}| = \sum_k \left|\lim_n a_{nk}\right|,$$

$$\textit{(9.3)} \quad \lim_n \sum_k |a_{nk} - \lim_n a_{nk}| = 0,$$

$$\textit{(9.4)} \quad \sum_k |a_{nk}| = 0 \textit{ converges uniformly in } n.$$

10 $A \in (c_0, c) \Leftrightarrow (1.1), (9.1)$.

11 $A \in (c, c) \Leftrightarrow (1.1), (9.1), (11.1)$. $A \in (c, c)$ *with* $\lim_n (Ax)_n = \lim_n x_n$ *for all* $x \in c$ $\Leftrightarrow (1.1), (5.1), (11.2)$.

(11.1) $\lim_n \sum_k a_{nk}$ *exists,*

(11.2) $\lim_n \sum_k a_{nk} = 1.$

12 $A \in (\ell_1, c) \Leftrightarrow (2.1), (9.1)$.

13 $A \in (\ell_p, c) \Leftrightarrow (3.1), (9.1)$.

14 $A \in (\ell_\infty, \ell_1) = (c, \ell_1) = (c_0, \ell_1) \Leftrightarrow (14.1) \Leftrightarrow (14.2) \Leftrightarrow (14.3) \Leftrightarrow (14.4)$.

(14.1) $\sup_{\substack{\text{N finite} \\ \text{K finite}}} \left| \sum_{n \in N} \sum_{k \in K} a_{nk} \right| < \infty,$

(14.2) $\sup_{\text{N finite}} \sum_k \left| \sum_{n \in N} a_{nk} \right| < \infty,$

(14.3) $\sup_{\text{K finite}} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty,$

(14.4) $\sum_n \left| \sum_{\substack{k \in K \\ K \subset \mathbb{N}_0}} a_{nk} \right| < \infty$ *converges uniformly in* K.

15 $A \in (\ell_1, \ell_1) \Leftrightarrow (15.1)$.

(15.1) $\sup_k \sum_n |a_{nk}| < \infty.$

16 $A \in (\ell_p, \ell_1) \Leftrightarrow (16.1)$.

(16.1) $\sup_{N \text{ finite}} \sum_k | \sum_{n \in N} a_{nk}|^q < \infty.$

17 $A \in (\ell_\infty, \ell_r) = (c, \ell_r) = (c_0, \ell_r) \Leftrightarrow (17.1) \Leftrightarrow (17.2)$. *There is r¿1,*

$$\text{(17.1)} \qquad \sup_{K\ \textit{finite}} \sum_n \left| \sum_{k \in K} a_{nk} \right|^r < \infty,$$

$$\text{(17.2)} \qquad \sum_n \left| \sum_{\substack{k \in K \\ K\ \textit{finite}}} a_{nk} \right|^r < \infty \ \textit{converges uniformly in}\ K.$$

18 $A \in (\ell_1, \ell_r) \Leftrightarrow (18.1)$.

$$\text{(18.1)} \qquad \sup_k \sum_n |a_{nk}|^r < \infty.$$

CHAPTER 5

HAUSDORFF MEASURE OF NONCOMPACTNESS OF OPERATORS

So far we measured the noncompactness of bounded subsets of metric spaces. Now we measure the noncompactness of operators. We are also going to apply our results to characterise the class of bounded compact linear operators from ℓ_1 into itself. Here we define the *measure of noncompactness of a linear operator* between Banach spaces. The definition is similar to that of the norm of a bounded linear operator between Banach spaces. We also study some properties of the measure of noncompactness of operators.

Definition 5.1. *Let κ_1 and κ_2 be measures of noncompactness on the Banach spaces X and Y, respectively.*

(a) *An operator $L : X \to Y$ is said to be (κ_1, κ_2)-bounded if*

$$L(Q) \in \mathcal{M}_Y \text{ for each } Q \in \mathcal{M}_X \tag{5.1}$$

and there exists a real k with $0 \leq k < 1$ such that

$$\kappa_2(L(Q)) \leq k\kappa_1(Q) \text{ for each } Q \in \mathcal{M}_X. \tag{5.2}$$

(b) *If an operator L is (κ_1, κ_2)-bounded then the number $||L||_{(\kappa_1,\kappa_2)}$ defined by*

$$||L||_{(\kappa_1,\kappa_2)} = \inf \{k \geq 0 : \kappa_2(L(Q)) \leq k\kappa_1(Q) \text{ for each } Q \in \mathcal{M}_X\} \tag{5.3}$$

is called (κ_1, κ_2)-operator norm of L, *or* (κ_1, κ_2)-measure of noncompactness of L, *or* simply measure of noncompactness of L. *If $\kappa_1 = \kappa_2 = \kappa$, we write $||L||_\kappa = ||L||_{(\kappa_1,\kappa_2)}$, for short.*

The first theorem is related to the Hausdorff measure of noncompactness of an operator.

Theorem 5.1. *[16, Theorem 2.25] Let X and Y be Banach spaces and $L \in \mathcal{B}(X,Y)$. Then we have*

$$||L||_{\chi} = \chi\left(L\left(S_X\right)\right) = \chi(L(\overline{B}_X)); \tag{5.4}$$

where $\overline{B}_X = \{x \in X : ||x|| \leq 1\}$ and $S_X = \{x \in X : ||x|| = 1\}$ are the closed unit ball and unit sphere in X.

Proof. We write $B = B_X$ and $S = S_X$, for short. Since $co(S) = B$ and $L(co(S)) = co(L(S))$, it follows from the last identity in Theorem 3.4 that

$$\chi\left(L\left(B\right)\right) = \chi\left(L\left(co\left(S\right)\right)\right) = \chi\left(co\left(L\left(S\right)\right)\right) = \chi\left(L\left(S\right)\right), \tag{5.5}$$

hence we have by (5.3) and Theorem 3.5 $\chi\left(L\left(B\right)\right) \leq ||L||_{\chi}$. Now we show $||L||_{\chi} \leq \chi\left(L\left(B\right)\right)$. Let $Q \in \mathcal{M}$ and $\{x_k : 1 \leq k \leq n\}$ be a finite r-net of Q. Then we have Q $\subset \cup_{k=1}^{n} B_r(x_k)$ and obviously

$$L(Q) \subset \bigcup_{k=1}^{n} L\left(B_r(x_k)\right). \tag{5.6}$$

It follows from (5.6), Lemma 3.2 and Theorem 3.4 that

$$\begin{aligned}
\chi\left(L\left(Q\right)\right) &\leq \chi\left(\bigcup_{k=1}^{n} L\left(B_r(x_k)\right)\right) \\
&\leq \max_{1\leq k\leq n} \chi\left(L\left(B_r(x_k)\right)\right) \\
&= \max_{1\leq k\leq n} \chi\left(\{x_k\} + L\left(B_r(0)\right)\right) \\
&= \chi\left(L\left(B_r(0)\right)\right) \\
&= \chi\left(r.L\left(B\right)\right) = |r|\chi\left(L\left(B\right)\right) = r\chi\left(L\left(B\right)\right)
\end{aligned}$$

hence $\chi\left(L\left(Q\right)\right) \leq \chi(Q)\chi\left(L\left(B\right)\right)$, and so $||L||_{\chi} \leq \chi\left(L\left(B\right)\right)$. □

We state the next result without proof.

Corollary 5.1. *[16, Corollary 2.26] Let X, Y and Z be Banach spaces, $\mathcal{C}(X,Y)$ denote the set of all compact operators from X into Y , $L \in \mathcal{B}(X,Y)$ and $\tilde{L} \in \mathcal{B}(Y,Z)$. Then $||.||_{\chi}$ is a seminorm on $\mathcal{B}(X,Y)$ and*

$$||L||_{\chi} = 0 \text{ if and only if } L \in \mathcal{C}(X,Y), \tag{5.7}$$

$$||L||_{\chi} \leq ||L||, \tag{5.8}$$

$$||L+K||_{\chi} = ||L||_{\chi}, \text{ for each } K \in \mathcal{C}(X,Y). \tag{5.9}$$

$$||\tilde{L}oL||_{\chi} \leq ||\tilde{L}||_{\chi}||L||_{\chi}. \tag{5.10}$$

5.1 AN APPLICATION

First, we characterise the bounded linear operators from the set of all absolutely convergent series into itself and determine the norm of such operators.

Theorem 5.2. *Let*

$$\ell_1 = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} |x_k| < \infty \right\}$$

denote the Banach space of all absolutely convergent series with

$$||x||_1 = \sum_{k=1}^{\infty} |x_k|.$$

We have $L \in B(\ell_1) = B(\ell_1, \ell_1)$ if and only if there exists an infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ of complex numbers such that

$$||A|| = \sup_k \sum_{n=0}^{\infty} |a_{nk}| < \infty \tag{5.11}$$

and

$$L(x) = A(x) \text{ for all } x \in \ell_1. \tag{5.12}$$

In this case, we have

$$||L|| = ||A||, \tag{5.13}$$

and the operator L uniquely determines the matrix $A = (a_{nk})_{n,k=1}^{\infty}$. The operator L is said to be given (defined) by the matrix A.

Proof. **(i)** First we assume $L \in \mathcal{B}(\ell_1) = \mathcal{B}(\ell_1, \ell_1)$. Since ℓ_1 has AK, $L \in \mathcal{B}(\ell_1)$ is given by a matrix $A \in (\ell_1, \ell_1)$ such that $L(x) = A(x)$ by Theorem 4.2. Since $A \in (\ell_1, \ell_1)$, we have $A_n \in \ell_1^\beta = \ell_\infty$ for each n. If we choose $x = e^{(k)}$, then we have

$$||L(e^{(k)})||_1 = \sum_{n=1}^{\infty} \left| \left(L(e^{(k)})\right)_n \right| = \sum_{n=1}^{\infty} \left| \left(A(e^{(k)})\right)_n \right| = \sum_{n=1}^{\infty} |a_{nk}| \leq ||L|| ||e^{(k)}||_1$$

that is,

$$||A|| = \sup_k \sum_{n=1}^{\infty} |a_{nk}| \leq ||L|| < \infty \text{ for all } k \tag{5.14}$$

and (5.11) holds. Furthermore, we have

$$\begin{aligned}
||L(x)||_1 = \sum_{n=1}^{\infty} |A_n(x)| &= \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} a_{nk} x_k \right| \\
&\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |a_{nk}| \, |x_k| \\
&\leq \sup_k \left(\sum_{n=1}^{\infty} |a_{nk}| \right) \sum_{k=1}^{\infty} |x_k| = ||A|| ||x||_1 \qquad \text{for all } x \in \ell_1,
\end{aligned}$$

$$||L(x)||_1 \leq ||A|| ||x||_1 \qquad \text{for all } x \in \ell_1, \tag{5.15}$$

and so $||L|| \leq ||A||$. This and (5.14) together yield (5.13).

(ii) Conversely let the condition in (5.11) hold. Then we obviously have

$$\sup_k |a_{nk}| < \infty \text{ for all } x \in \mathbb{N},$$

that is, $A_n \in \ell_\infty$ for all $n \in \mathbb{N}$. Let $x \in \ell_1$. Then we obtain as in (5.15) $A(x) \in \ell_1$, whence $A \in (\ell_1, \ell_1)$. We define the linear operator $L : \ell_1 \to \ell_1$ by (5.12). Then it follows that $L \in \mathcal{B}(\ell_1)$.

□

Now we evaluate the Hausdorff measure of noncompactness of an operator $L \in B(\ell_1)$.

Theorem 5.3. *Let $L \in \mathcal{B}(\ell_1)$. Then L is given by an infinite matrix A, and we have*

$$||L||_\chi = \lim_{m \to \infty} \left(\sup_k \sum_{n=m}^{\infty} |a_{nk}| \right). \tag{5.16}$$

Proof. We write $S = S_{\ell_1}$ for the unit sphere in ℓ_1. It follows from Theorems 3.6, 3.5 and 5.2 that

$$||L||_\chi = \chi\left(L\left(S\right)\right) = \lim_{m\to\infty}\left(\sup_{x\in S}\sum_{n=m}^{\infty}\Big|\sum_{k=1}^{\infty}a_{nk}x_k\Big|\right). \tag{5.17}$$

The limit in (5.16) obviously exists. From

$$\begin{aligned}\sum_{n=m}^{\infty}\Big|\sum_{k=1}^{\infty}a_{nk}x_k\Big| \le \sum_{n=m}^{\infty}\sum_{k=1}^{\infty}|a_{nk}x_k| &= \sum_{k=1}^{\infty}\left(\sum_{n=m}^{\infty}|a_{nk}|\right)|x_k| \\ &\le \left(\sup_k\sum_{n=m}^{\infty}|a_{nk}|\right)||x||_1\end{aligned}$$

and we obtain

$$\sup_{x\in S}\sum_{n=m}^{\infty}\Big|\sum_{k=1}^{\infty}a_{nk}x_k\Big| \le \sup_k\sum_{n=m}^{\infty}|a_{nk}| \text{ for all } x\in\ell_1. \tag{5.18}$$

To prove the converse inequality, we choose $x = e^{(k)} \in \ell_1$ for $k \in \mathbb{N}$. Since $L(e^{(k)}) = A^k = (a_{nk})_{n=0}^{\infty}$, Theorem 3.6 implies

$$\begin{aligned}\chi\left(\left\{L(e^{(k)}) : k=1,2,\ldots,\right\}\right) &= \lim_{m\to\infty}\left(\sup_{x\in\left\{L(e^{(k)}):k=1,2,\ldots,\right\}}\sum_{n=m}^{\infty}|a_{nk}|\right) \\ &= \lim_{m\to\infty}\left(\sup_k\sum_{n=m}^{\infty}|a_{nk}|\right) \le \chi\left(L\left(S\right)\right).\end{aligned}$$

This and inequality (5.18) together yield (5.16). □

Theorem 5.3 and (5.7) in Corollary 5.1 yield the characterisation of the class $C(\ell_1) = C(\ell_1, \ell_1)$.

Corollary 5.2. *Let $L \in \mathcal{B}(\ell_1)$ be given by an infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$. Then we have $A \in C(\ell_1)$ if and only if*

$$\lim_{m\to\infty}\left(\sup_k\sum_{n=m}^{\infty}|a_{nk}|\right) = 0.$$

CHAPTER 6

MATRIX DOMAINS

In this chapter, We shall characterize matrix transformations between some spaces and apply the Hausdorff measure of noncompactness to give necessary and sufficient conditions for these matrix maps to be compact operators.

Definition 6.1. *Let X be a set of sequences and A an infinite matrix. Then the set*

$$X_A = \{x \in \omega : A(x) \in X\}$$

is called the (ordinary) matrix domain of A. *In the special case where $X = c$, the set c_A is called* convergence domain of A*.*

Lemma 6.1. *[16, Lemma 3.2] Let X be a linear space, $(Y, ||.||)$ a normed space and $T : X \to Y$ a linear one-to-one map. Then X becomes a normed space with $||x||_X = ||T(x)||$. If, in addition, Y is a Banach space and T is onto Y , then $(X, ||.||_X)$ is a Banach space.*

Theorem 6.1. *Let T be a triangle and $(X, ||.||)$ be a BK space. Then X_T is a BK space with $||x||_T = ||T(x)||$.*

Proof. We define the map $L_T : X_T \to X$ by $L_T(x) = T(x)$ for all $x \in X_T$. Then L_T is linear, one-to-one, since T is a triangle, and onto X, since $X_T = L_T^{-1}(X)$ and L_T is one-to-one. By Lemma 6.1, X_T is a Banach space. We show that the coordinates are continuous in X_T . Let $x^{(n)} \to x$ in X_T . Then $y_k^{(n)} = T_k(x^{(n)}) \to y_k = T_k(x)$, since X is a BK space. Let S be the inverse of T, also a triangle. Then $x_k^{(n)} = \sum_{j=0}^{k} s_{kj} y_j^{(n)} \to \sum_{j=0}^{k} s_{kj} y_j = x_k$, that is, $P_k(x^{(n)}) \to P_k(x)$. This shows that the coordinates are continuous on X_T . □

As a special case of Theorem 6.1, we obtain,

Corollary 6.1. *[23, Theorem 4.3.13] Let T be a triangle. Then c_T is a BK space with $||x||_{T,\infty} = ||T(x)||_\infty$.*

Theorem 6.2. *[23, Theorem 4.3.14] If X is a closed subspace of Y, then X_A is a closed subspace of Y_A.*

6.1 MATRIX TRANSFORMATIONS INTO MATRIX DOMAINS

In this section, we shall show that, for triangles T, the characterizations of the class (X, Y_T) can be reduced to that of (X, Y).

Theorem 6.3. *[21, Theorem 1] Let T be a triangle.*

(a) *Then, for arbitrary subsets X and Y of ω, $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.*

(b) *Further, if X and Y are BK spaces and $A \in (X, Y_T)$, then*

$$||L_A|| = ||L_B||. \tag{6.1}$$

Proof. Let $x \in X$. Since $A_n \in X^\beta$ for all $n = 0, 1, \ldots$, we have $x \in \omega_A$. Further $T_n \in \phi(n = 0, 1, \ldots)$ because T is a triangle. Therefore, $(TA)(x) = T(A(x))$ (cf. [23, Theorem 1.4.4, p. 8]).

(a) **$\Rightarrow$:** Let $A \in (X, Y_T)$ and $x \in X$. If we consider $A(x) \in Y_T$ and $T(A(x)) = (TA)(x)$, $(TA)(x) \in Y$. So $TA \in (X, Y)$

$\Leftarrow$: Let $B = TA \in (X, Y)$ and $x \in X$. Since $(TA)(x) = T(A(x))$ and the assumption, $T(A(x)) \in Y$, that is, $A(x) \in Y_T$. So $A \in (X, Y_T)$.

(b) Let $A \in (X, Y_T)$. Since Y is a BK space and T a triangle, Y_T is a BK space with

$$||y||_{Y_T} = ||T(y)||_Y \quad (y \in Y_T) \tag{6.2}$$

by Theorem 6.1. Thus A is continuous by Theorem 2.2 and consequently

$$\|L_A\| = \sup\{\|L_A(x)\|_{Y_T} : \|x\| = 1\} = \sup\{\|A(x)\|_{Y_T} : \|x\| = 1 < \infty\}. \quad (6.3)$$

Further, since B is continuous,

$$||L_B|| = \sup\{||L_B(x)||_Y : ||x|| = 1\} = \sup\{||B(x)||_Y : ||x|| = 1 < \infty\}. \quad (6.4)$$

If we define $B(x) = (TA)(x) = T(A(x))$, (6.1) follows from (6.2), (6.3) and (6.4).

Definition 6.2. *Let $q = (q_k)$ be a sequence of non-negative real numbers with $q_0 > 0$ and write*

$$Q_n = \sum_{k=0}^{n} q_k \text{ for all } n \in \mathbb{N}. \quad (6.5)$$

Then the Riesz mean with respect to the sequence $q = (q_k)$ is defined by the matrix $\bar{N}_q = \{(\bar{N}_q)_{nk}\}$ with

$$(\bar{N}_q)_{nk} = \begin{cases} \frac{q_k}{Q_n}, & 0 \le k \le n, \\ \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

It is known that ℓ_p is a BK space with the natural norm $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p}$ $(p \ge 1)$, so $(\ell_p)_{\overline{N}_q}$ is a BK space by Theorem 6.1.

We know that there exist a unique inverse of any triangle matrix. Let $x \in (\ell_p)_{\bar{N}_q}$, so $y = \bar{N}_q(x) \in \ell_p$ and

$$y_n = \left(\bar{N}_q(x)\right)_n = 1/Q_n \sum_{k=0}^{n} q_k x_k. \quad (6.6)$$

If we consider $Q_n y_n - Q_{n-1} y_{n-1}$, we obtain

$$x_n = 1/q_n \sum_{k=n-1}^{n} (-1)^{n-k} Q_k y_k. \quad (6.7)$$

We can write $\bar{R}_q = \{(\bar{R}_q)_{nk}\}$

$$(\bar{R}_q)_{nk} = \begin{cases} (-1)^{n-k} \frac{Q_k}{q_n}, & n-1 \le k \le n, \\ \\ 0, & others \end{cases}$$

for all $n, k \in \mathbb{N}$. Strictly speaking, we have implicitly for all $x \in (\ell_p)_{\bar{N}_q}$

$$\begin{aligned}
\left(\bar{R}_q \cdot \bar{N}_q\right)(x) &= \bar{R}_q\left(\bar{N}_q(x)\right) \\
&= \bar{R}_q\left(1/Q_n \sum_{k=0}^{n} q_k x_k\right) \\
&= \sum_{k=n-1}^{n} (-1)^{n-k} \frac{Q_k}{q_n}\left(1/Q_k \sum_{i=0}^{k} q_i x_i\right) \\
&= \frac{1}{q_n}\left(\sum_{i=0}^{n} q_i x_i - \sum_{i=0}^{n-1} q_i x_i\right) \\
&= x_n = I(x),
\end{aligned}$$

also we have for each $y \in \ell_p$

$$\begin{aligned}
\left(\bar{N}_q \cdot \bar{R}_q\right)(y) &= \bar{N}_q\left(\bar{R}_q(y)\right) \\
&= \bar{N}_q\left(1/q_n \sum_{k=n-1}^{n} (-1)^{n-k} Q_k y_k\right) \\
&= \frac{1}{Q_n} \sum_{k=0}^{n} q_k\left(1/q_k \sum_{i=k-1}^{k} (-1)^{k-i} Q_i y_i\right) \\
&= \frac{1}{Q_n} \sum_{k=0}^{n} (Q_k y_k - Q_{k-1} y_{k-1}) \\
&= y_n = I(y),
\end{aligned}$$

that is,

$$\bar{R}_q \cdot \bar{N}_q = \bar{N}_q \cdot \bar{R}_q = I. \tag{6.8}$$

Therefore, $\left(\bar{N}_q\right)^{-1} = \bar{R}_q$.

Theorem 6.4. *Since ℓ_p is BK space with AK, $\left(\bar{R}_q\left(e^{(n)}\right)\right)_{n=0}^{\infty}$ is a basis for $(\ell_p)_{\bar{N}_q}$ by [22, Theorem 2.2]. [24, Corollary 2.5(a) and (2.1)] yield a unique representation for each $x \in (\ell_p)_{\bar{N}_q}$.*

If x and y are sequences and X and Y are subsets of ω then we write $xy = (x_k y_k)_{k=0}^{\infty}$, $x^{-1}*Y = \{a \in \omega : ax \in Y\}$ and $M(X,Y) = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$ for the multiplier space of X and Y.

We define the matrics S, Δ^+ by $S_{nk} = 1$ $(0 \le k \le n)$, $S_{nk} = 0$ $(k > n)$, $\Delta^+_{nn} = 1$, $\Delta^+_{n,n+1} = -1$ and $\Delta^+_{nk} = 0$ otherwise for all n, and use the convention that any term with a negative subscript is equal to zero.

By $\mathcal{U}$, we denote the set of all sequences u with $u_k \neq 0$ for all k, and we write $1/u = (1/u_k)_{k=0}^{\infty}$. Furthermore, we know that $(u^{-1} * \ell_p)^\dagger = \left(\frac{1}{u}\right)^{-1} * [(\ell_p)]^\dagger$ for $\dagger = \alpha, \beta, \gamma$; $[(\ell_p)_S]^\alpha = \left(\ell_p^\alpha\right)_{\Delta^+} \cap \ell_p^\alpha$, $[(\ell_p)_S]^\beta = \left(\ell_p^\beta\right)_{\Delta^+} \cap M(\ell_p, c_0)$ and $[(\ell_p)_S]^\gamma = \left(\ell_p^\gamma\right)_{\Delta^+} \cap M(\ell_p, \ell_\infty)$. These identities are valid when we use $u * \ell_p$ instead of ℓ_p for $u \in \mathcal{U}$. One can also find their general forms in [22, Lemma 2.1, Corollary 2.1 and 2.2.].

If we put $u = 1/Q$, $v = q$, hence $b = Q\Delta^+(a/q)$ and $d = Qa/q$ for $a \in \omega$, then we immediately obtain the following theorem from [22, Theorem 3.1].

Theorem 6.5. *Let $q = (q_k)$ be a sequence of non-negative real numbers with $q_0 > 0$, $Q = (Q_n)$ such that $Q_n = \sum_{k=0}^{n} q_k$ for all n, $1 \leq p < \infty$ and r be the conjugate number of p, that is, $r = \infty$ and $r = p/p-1$ for $1 < p < \infty$. Then we have*

$$\left[(\ell_p)_{\bar{N}_q}\right]^\alpha = \{a = (a_k) \in \omega : b \in \ell_r \text{ and } d \in \ell_r\}, \tag{6.9}$$

$$\left[(\ell_p)_{\bar{N}_q}\right]^\beta = \left[(\ell_p)_{\bar{N}_q}\right]^\gamma = \{a = (a_k) \in \omega : b \in \ell_r \text{ and } d \in \ell_\infty\}. \tag{6.10}$$

If $X \supset \phi$ is a BK-space and $a = (a_k) \in \omega$, then we define

$$\|a\|_{X^\beta} = \|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=0}^{\infty} a_k x_k \right| \tag{6.11}$$

provided the expression on the right hand side exists and is finite. Here, $\|.\|_{X^\beta}$ denotes the natural norm on the dual space X^β.

Theorem 6.6. *[22, Theorem 3.3] Let $1 \leq p < \infty$ and $r = p/(p-1)$. Then the necessary and sufficient conditions for the operator A from $(\ell_p)_{\bar{N}_q}$ into ℓ_∞ c_0, c and ℓ_1 can be read from the following table:*

To/From	ℓ_∞	c_0	c	ℓ_1
$(\ell_p)_{\bar{N}_q}$	*1*	*2*	*3*	*4*

where

1 *(1.1), (1.2) where*

(1.1) $\sup_k |Q_k a_{nk}/q_k| < \infty$ *for all* n,

$$(1.2)\quad \begin{cases} \sup_n \sum_{k=0}^{\infty} |Q_k(a_{nk}/q_k - a_{n,k+1}/q_{k+1})|^r < \infty \ (1 < p < \infty) \\ \sup_{n,k} |Q_k(a_{nk}/q_k - a_{n,k+1}/q_{k+1})| < \infty \ (p = 1) \end{cases}$$

2 *(1.1), (1.2), (2.1) where*

(2.1) $\lim_{n\to\infty} |Q_k(a_{nk}/q_k - a_{n,k+1}/q_{k+1})| = 0$ *for each* k

3 *(1.1), (1.2), (3.1) where*

(3.1) $\lim_{n\to\infty} |Q_k(a_{nk}/q_k - a_{n,k+1}/q_{k+1})| = \alpha_k$ *for each* k

4 *(1.1), (4.1) where*

$$(4.1)\quad \begin{cases} \sup_N \sum_{k=0}^{\infty} \left| Q_k \sum_{n\in N} (a_{nk}/q_k - a_{n,k+1}/q_{k+1}) \right|^r < \infty \ (1 < p < \infty) \\ \sup_k |Q_k| \sum_{n\in N} |a_{nk}/q_k - a_{n,k+1}/q_{k+1}| < \infty \ (p = 1). \end{cases}$$

where the supremum is taken over all finite subset N of $\mathbb{N}$

We obtain the following Lemma as an immediate consequence of [22, Theorem 2.4(2.11)].

Lemma 6.2. *Let* $1 \le p < \infty$ *and* r *be the conjugate number. If* $a = (a_k) \in \left[(\ell_p)_{\bar{N}_q}\right]^{\beta}$, *then*

$$\begin{aligned} \sum_{k=0}^{\infty} a_k x_k &= \sum_{k=0}^{\infty} \Delta_k(a/q) \sum_{j=0}^{k} q_j x_j \\ &= \sum_{k=0}^{\infty} \left[Q_k \Delta_k(a/q)\right] (\bar{N}_q) x \end{aligned}$$

for all $x = (x_k) \in (\ell_p)_{\bar{N}_q}$.

For any infinite matrix $A = (a_{nk})$, we define the associated matrix $\tilde{A} = (\tilde{a}_{nk})$ by

$$\widehat{a}_{nk} = \Delta\left(\frac{a_{nk}}{q_k}\right)Q_k = \left(\frac{a_{nk}}{q_k} - \frac{a_{n,k+1}}{q_{k+1}}\right)Q_k \tag{6.12}$$

$\forall n, k \in \mathbb{N}$.

We obtain the following Lemma as an immediate consequence of [25, Theorems 3.2 and 3.4].

Lemma 6.3. *Let Y be an arbitrary subset of ω, $1 \le p < \infty$ and $A \in \left((\ell_p)_{\bar{N}_q}, Y\right)$. Then $\widehat{A} \in (\ell_p, Y)$, where the entries of the matrix $\widehat{A}$ are given by (6.12), and*

$$Ax = \widehat{A}(\overline{N}_q x) \text{ for all } x = (x_k) \in (\ell_p)_{\bar{N}_q}.$$

Futhermore, it follows by [25, Theorem 3.6], if $A \in \left((\ell_p)_{\bar{N}_q}, Y\right)$ then

$$\|L_A\| = \|L_{\widehat{A}}\|. \tag{6.13}$$

Theorem 6.7. *[27, Theorem 2.8] Let $1 \le p < \infty$ and r be the conjugate number of p, that is, $r = \infty$ for $p = 1$ and $r = p/p-1$ for $0 < r < 1$.*

(a) *Let $Y = c_0, c, \ell_\infty$. If $A \in \left((\ell_p)_{\bar{N}_q}, Y\right)$ then we put*

$$(\star) \qquad \|A\|_{\left((\ell_p)_{\bar{N}_q},\infty\right)} = \sup_n \|\widehat{A}_n\|_{\ell_q} = \begin{cases} \sup_n \left(\sum_{k=0}^{\infty} |\widehat{a}_{nk}|^q\right)^{1/q} & (1 < p < \infty) \\ \\ \sup_{n,k} |\widehat{a}_{nk}| & (p = 1) \end{cases}$$

Then we have

$$\|L_A\| = \|A\|_{\left((\ell_p)_{\bar{N}_q},\infty\right)}.$$

(b) *Let $Y = \ell_1$. If $A \in \left((\ell_p)_{\bar{N}_q}, \ell_1\right)$ then we put*

$$\|A\|_{\left((\ell_p)_{\bar{N}_q},1\right)} = \sup_N \left\|\sum_{n\in N} \widehat{A}_n\right\|_{\ell_q} = \sup_N \left(\sum_{k=0}^{\infty} \left|\sum_{n\in N} \widehat{a}_{nk}\right|^q\right)^{1/q} \quad (1 < p < \infty).$$

Then we have

$$\|A\|_{\left((\ell_p)_{\bar{N}_q},1\right)} \le \|L_A\| \le 4 \cdot \|A\|_{\left((\ell_p)_{\bar{N}_q},1\right)}.$$

Proof. Since ℓ_p is a BK space with AK, $A \in \left((\ell_p)_{\bar{N}_q}, Y\right)$ implies $\widehat{A} \in (X, Y)$ by Lemma 6.3, so $(\star)$ holds in each case.

(a) If $Y = c_0, c, \ell_\infty$, then we have $\|L_A\| = \|L_{\widehat{A}}\| = \sup_n \|A_n\|^*_{\ell_p}$ by [16, Theorem 1.23]. Now $(\star)$ follows from the definition of norms $\|.\|_{\left((\ell_p)_{\bar{N}_q}, \infty\right)}$ and the fact that $\|.\|^*_{\ell_p} = \|.\|_r$.

(b) Let $Y = \ell_1$. Then again we have $\|L_A\| = \|L_{\widehat{A}}\|$ and by [26, Proposition 4.3], we obtain with

$$\|\widehat{A}\|_{(\ell_p,\ell_1)} = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left\| \sum_{n \in N} \widehat{A}_n \right\|^*_{\ell_p} < \infty,$$

$$\|\widehat{A}\|_{(\ell_p,\ell_1)} \leq \|L_{\widehat{A}}\| \leq 4 \cdot \|\widehat{A}\|_{(\ell_p,\ell_1)}.$$

□

We obtain the following Theorem as an immediate consequence of [27, Corollary 3.6].

Theorem 6.8. *Let $1 \leq p < \infty$ and r be the conjugate number of p. Then, we have the following*

(a) *If $A \in \left((\ell_p)_{\bar{N}_q}, c_0\right)$, then*

$$\|L_A\|_\chi = \lim_{r \to \infty} \left(\sup_{n > r} \|\widehat{A}_n\|_{\ell_r} \right). \tag{6.14}$$

(b) *If $A \in \left((\ell_p)_{\bar{N}_q}, c\right)$, then*

$$\frac{1}{2} \lim_{r \to \infty} \left(\sup_{n \geq r} \|\widehat{A}_n - \widehat{\alpha}\|_{\ell_r} \right) \leq \|L_A\|_\chi \leq \lim_{r \to \infty} \left(\sup_{n \geq r} \|\widehat{A}_n - \widehat{\alpha}\|_{\ell_r} \right), \tag{6.15}$$

where $\tilde{\alpha} = (\tilde{\alpha}_k)_{k=0}^\infty$ with $\tilde{\alpha}_k = \lim_{n \to \infty} \tilde{a}_{nk}$ for all k.

(c) *If $A \in \left((\ell_p)_{\bar{N}_q}, \ell_1\right)$, then*

$$\lim_{m \to \infty} \left(\sup_{N_m} \left\| \sum_{n \in N_m} \widehat{A}_n \right\|_{\ell_r} \right) \leq \|L_A\|_\chi \leq 4 \cdot \lim_{m \to \infty} \left(\sup_{N_m} \left\| \sum_{n \in N_m} \widehat{A}_n \right\|_{\ell_r} \right). \tag{6.16}$$

Proof. **(a)** This follows from Lemma 6.3, (5.4), (3.20) and Theorem 6.7 (a).

(b) This follows from Lemma 6.3, (5.4), (3.26) and Theorem 6.7 (a).

(a) This follows from Lemma 6.3, (5.4), (3.20) and Theorem 6.7 (b).

□

We obtain the following theorem from (5.7) and Theorem 6.2.

Corollary 6.2. *Let $1 \leq p < \infty$ and r be the conjugate number of p. Then, we have the following*

(a) *If $A \in \left((\ell_p)_{\bar{N}_q}, c_0\right)$, then*

$$L_A \textit{ is compact if and only if } \lim_{r\to\infty}\left(\sup_{n>r}\|\widehat{A}_n\|_{\ell_r}\right) = 0.$$

(b) *If $A \in \left((\ell_p)_{\bar{N}_q}, c\right)$, then*

$$L_A \textit{ is compact if and only if } \lim_{r\to\infty}\left(\sup_{n\geq r}\|\widehat{A}_n - \widehat{\alpha}\|_{\ell_r}\right) = 0.$$

where $\tilde{\alpha} = (\tilde{\alpha}_k)_{k=0}^{\infty}$ with $\tilde{\alpha}_k = \lim_{n\to\infty} \tilde{a}_{nk}$ for all k.

(c) *If $A \in \left((\ell_p)_{\bar{N}_q}, \ell_1\right)$, then*

$$L_A \textit{ is compact if } \lim_{m\to\infty}\left(\sup_{N_m}\left\|\sum_{n\in N_m}\widehat{A}_n\right\|_{\ell_r}\right) = 0.$$

CHAPTER 7

CONCLUSION

As a summary, a concise, self-contained and comprehensive outline of the modern functional analytic theories of FK, BK, AK and AD spaces, and of measures of noncompactness are examined. Some applications of the theories to the characterizations of the classes of linear operators between the classical sequence spaces are given. Identities or estimates for the Hausdorff measure of compactness of operator between some sequence spaces, and characterizations of compact operators are established. Finally, a few applications to matrix domains of triangles are considered.

CHAPTER 8

APPENDIX

8.1 INEQUALITIES

Theorem 8.1. *(Hölder's inequality)*

Let $1 < p < \infty$, $q = p/(p-1)$ *and* $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_n \in \mathbb{C}$ *. Then*

$$\sum_{k=0}^{n} |x_k y_k| \leq \left(\sum_{k=0}^{n} |x_k|^p \right)^{1/p} \left(\sum_{k=0}^{n} |y_k|^q \right)^{1/q} \quad \textit{(Hölder's inequality)} .$$

Or series if $x \in \ell_p$ *and* $y \in \ell_q$ *then* $xy = (x_k y_k)_{k=0}^{\infty} \in \ell_1$ *and*

$$||xy||_1 \leq ||x||_p ||y||_q.$$

Theorem 8.2. *(Minkowski's inequality)*

Let $1 \leq p < \infty$ *and* $q = p/(p-1)$ *and* $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_n \in \mathbb{C}$ *. Then*

$$\left(\sum_{k=0}^{n} |x_k + y_k|^p \right)^{1/p} \leq \left(\sum_{k=0}^{n} |x_k|^p \right)^{1/p} \left(\sum_{k=0}^{n} |y_k|^p \right)^{1/p} \quad \textit{(Minkowski's inequality)} .$$

Or series if $x, y \in \ell_p$ *then* $x + y \in \ell_p$ *and*

$$||x+y||_p \leq ||x||_p + ||y||_p.$$

Theorem 8.3. *(Jensen's inequality)*

Let $x_0, x_1, \ldots, x_n \in \mathbb{C}$ *. Then*

$$\sum_{k=0}^{n} |x_k|^{p'} < \infty \textit{ for some } p' > 0,$$

then

$$\sum_{k=0}^{n} |x_k|^p \textit{ is a decreasing function in } p > p'.$$

8.2 THE CLOSED GRAPH THEOREM AND THE BANACH-STEINHAUS THEOREM

In this appendix, we collect the results from Functional Analysis needed in the previous sections

Theorem 8.4. *(Closed graph lemma)*
Any continuous map into a Hausdorff space has closed graph ([11, Theorem 11.1.1, p. 195]).

Theorem 8.5. *(Closed graph theorem)*
If X and Y are Fréchet spaces and $f : X \to Y$ is a linear map with closed graph, then f is continuous [11, Theorem 11.2.2, p. 200].

Theorem 8.6. *(Banach-Steinhaus theorem)*
Let $(f_n)_{n=0}^{\infty}$ be a pointwise convergent sequence of continuous linear functionals on a Fréchet space X. Then f defined by

$$f(x) = \lim_{n\to\infty} f_n(x) \text{ for all } x \in X.$$

is continuous [11, Corollary 11.2.4, p. 200].

8.3 THE HEINE-BOREL THEOREM AND THE HAHN-BANACH THEOREM

Theorem 8.7. *(Heine-Borel Theorem)*
In n-dimensional Euclidean space, a subset of the space is compact if and only if it is closed and bounded.

Theorem 8.8. *[23, 3.0.1](Hahn-Banach theorem) Let X be a subspace of a linear topological space Y and f be a linear functional on X which is continuous in the relative topology of Y. Then f can be extended to a continuous linear functional on Y.*

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