T. R.
VAN YUZUNCU YIL UNIVERSITY
INSTITUTE OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF STATISTICS

ANALYSIS OF SOLUTIONS OF STOCHASTIC EVOLUTION EQUATIONS

Ph.D THESIS

PREPARED BY: Mohanad ALALOUSH
SUPERVISOR: Assist. Prof. Hatice TAŞKESEN

VAN-2020

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ACCEPTANCE and APPROVAL PAGE

This thesis entitled "**Analysis of Solutions of Stochastic Evolution Equations**" presented by Mohanad ALALOUSH under supervision of Assist. Prof. Hatice TAŞKESEN in the Department of Statistics has been accepted as a Ph.D. thesis according to Legislations of Graduate Higher Education on 21/12/2020 with unanimity of votes members of jury.

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This thesis has been approved by the committee of The Institute of Natural and Applied Science on ..……../........../…..….. with decision number ....

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THESIS STATEMENT

All information presented in the thesis obtained in the frame of ethical behavior and academic rules. In addition all kinds of information that does not belong to me have been cited appropriately in the thesis prepared by the thesis writing rules.

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Mohanad Alaloush

ABSTRACT

ANALYSIS OF SOLUTIONS OF STOCHASTIC EVOLUTION EQUATIONS

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Ph.D. Thesis Department of Statistics
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Nonlinear evolution equations are equations that contain the time variable t as an argument, appearing not only in many fields of mathematics but also in other branches of science such as physics, mechanics, and materials science. Since the deterministic evolution equations are insufficient in the modeling of physical phenomena, a term including the effect of uncertainty is usually added to the deterministic evolution equations. In this thesis, we investigate the effect of these terms, i.e. noise, on the solutions of some evolution equations. For this purpose, we use the Hermite transform and Galilean transform to obtain the stochastic equations deterministic counterparts and then use some analytical methods to obtain the solutions. The first chapter of the thesis includes a motivating example explaining why stochastic differential equations are needed. The second chapter summarizes the literature review. The third chapter contains the basic concepts, definitions, and preliminaries of the methods that are used in the thesis. In the fourth chapter, analytical solutions of stochastic KdV-Burgers, stochastic KdV, stochastic Burgers, stochastic Kuramoto-Sivashinsky and stochastic Kawahara equations are obtained by using Galilean transform and tanh, extended tanh methods. Moreover, the solutions of a stochastic Wick-type extended KdV equation are found by using Hermite transform and Jacobi elliptic functions. The illustrations of some solutions are given to see the effect of noise apparently.

Keywords: Galilean transform, Hermite transform, Jacobi elliptic functions, KdV-Type equations, Stochastic evolution equations.

ÖZET

STOKASTİK EVOLÜSYON DENKLEMLERİNİN ÇÖZÜMLERİNİN ANALİZİ

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Doğrusal olmayan evolüsyon denklemleri, sadece matematiğin birçok alanında değil, aynı zamanda fizik, mekanik ve malzeme bilimi gibi diğer bilim dallarında da ortaya çıkan, argüman olarak t zaman değişkenini içeren denklemlerdir. Fiziksel olayların modellenmesinde deterministik evolüsyon denklemleri yetersiz olduğundan, deterministik evolüsyon denklemlerine genellikle belirsizliğin etkisini içeren bir terim eklenir.Bu tezde bu terimlerin, yani gürültünün bazı evolüsyon denklemlerinin çözümleri üzerindeki etkisini araştırıldı. Bu amaçla, çalışılan stokastik denklemlerin deterministik karşılıklarını elde etmek için Hermite dönüşümü ve Galilean dönüşümü kullanıldı ve daha sonra bazı analitik yöntemlerle çözümler elde edildi. Tezin ilk bölümü, stokastik diferansiyel denklemlere neden ihtiyaç duyulduğunu açıklayan motive edici bir örnek içermektedir. İkinci bölüm literatür taramasını özetlemektedir. Üçüncü bölümü, tezde kullanılan kavramlar, tanımlar ve kullanılan yöntemlerle ilgili ön bilgileri, dördüncü bölümü ise, Galilean dönüşümü ve tanh, genişletilmiş tanh yöntemleri kullanılarak stokastik KdV-Burgers, stokastik KdV, stokastik Burgers, stokastik Kuramoto-Sivashinsky ve stokastik Kawahara denklemleri için elde edilmiş analitik çözümleri içermektedir. Ayrıca, stokastik Wick tipi bir genişletilmiş KdV denkleminin çözümleri Hermite dönüşümü ve Jacobi eliptik fonksiyonları kullanılarak bulunmuştur. Gürültünün etkisinin görülebilmesi için bazı çözümlerin grafiklerine de yer verilmiştir.

Anahtar kelimeler: Stochastic evolüsyon denklemler, KdV-Tipli denklemler, Hermite dönüşümü, Galilean dönüşümü, Jacobi eliptik fonksiyonlar.

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Mohanad ALALOUSH

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1. INTRODUCTION

Nonlinear evolution equations are partial differential equations that contain the time variable t as an independent variable and arise not only in many fields of mathematics but also in other disciplines such as physics, mechanics, and materials science. The Navier-Stokes and Euler equations are just a few of the nonlinear evolution equations that arise in fluid mechanics, reaction-diffusion equations in heat transfer and biological sciences, Klein-Gordon and Schrödinger equations in quantum mechanics, and Cahn-Hilliard equations in material science. Since the deterministic models ignore the effects of many small perturbations, stochastic equations adapt better to events. For example, in the modeling of waves on the surface of shallow water, an unstable pressure may affect the surface of fluid or the layer's bottom may not be flat, so, adding a stochastic term containing these effects creates a more realistic and meaningful model. One way to make these unrealistic deterministic models more meaningful is to take the average of some parameters, but this is not a satisfactory way. The following example given in (Holden, Øksendal, Ubøe, and Zhang, 2010) may explain this situation very well. Assume that a porous and dry (heterogeneous and isotropic) rock is injected a fluid with the rate of injection $f(t,x)$ (at the point $x \in R^3$ and at time t). Fluid flow on the surface of the rock can be defined mathematically as in the following way:

Let us denote the saturation and pressure of the fluid by $\varphi(t,x)$ and $p_t(x)$, respectively, at (t,x). Suppose that either we have full saturation $\varphi_0(x) > 0$ at time t, or the point x is dry at time t, that is, $\varphi(t,x) = 0$. Let the wet region at time t be defined by Ω_t

$$\Omega_t = \{x; \quad \varphi(t,x) = \varphi_0(x)\}.$$

Then by combining the continuity equation and Darcy's law, which is describing the flow of a fluid on a porous medium, we conclude a moving boundary problem for the unknowns Ω_t and $p_t(x)$

$$\begin{gathered} \mathrm{div}\big(k(x)\nabla p_t(x)\big) = -f_t(x); x \in \Omega_t \\ p_t(x) = 0; x \in \partial\Omega_t \\ \varphi_0(x) \cdot \frac{d}{dt}(\partial\Omega_t) = -k(x)N^T(x)\nabla p_t; x \in \partial\Omega_t, \end{gathered} \tag{1.0.1}$$

where the gradients and divergence are taken with respect to x, the density and viscosity are taken as 1, and $N(x)$ is the outer unit normal of $\partial\Omega_t$ at x. We assume that the initial wet region Ω_0 is known and that $supp f_t \subset \Omega_t$ for all t. In the third equation of (1.0.1), $k(x) \geq 0$ denotes the permeability, i.e., the ability to allow fluids to pass through the rock at point x which is defined as the proportionality constant in Darcy's law

$$q_t(x) = -k(x)\nabla p_t(x),$$

where $q_t(x)$ is the flow velocity of the fluid. In a typical porous rock, $k(x)$ is fluctuating in an irregular, unpredictable way. Considering the difficulty of solving (1.0.1) for such a permeability function $k(x)$, one may be tempted to replace $k(x)$ by its x -average $\bar{k}$ (constant) and solve this system instead. This, however, turns out to give a solution that does not describe what actually happens! For example, if we let $f_t(x) = \delta_0(x)$ be a point source at the origin and choose Ω_0 to be an open unit ball centered at 0, then it is easy to see (by symmetry) that the system (1.0.1) with $k(x) \equiv \bar{k}$ will give the solution $\{\Omega_t\}_{t\geq 0}$ consisting of an increasing family of open balls centered at 0.

Actual experiments with fluid flow in porous rocks show that such a solution is, in fact, a fractal. The following figures illustrates the averaged permeability constant, and the fractal nature of the fluid diffusion through a rock.

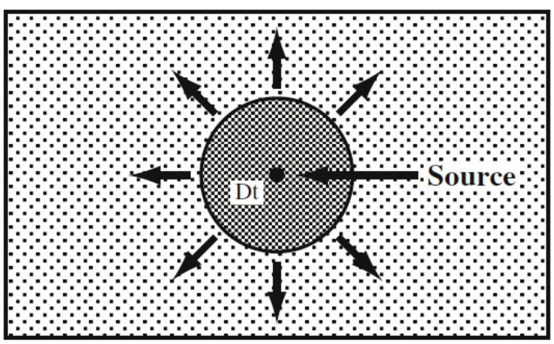

Figure 1.1. A constant permeability $\bar{k}$ leads to solutions consisting of expanding balls centered at the injection hole.

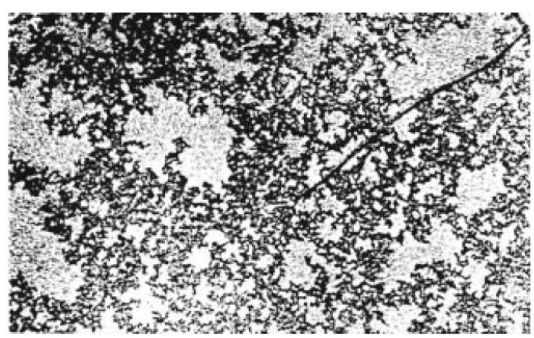

Figure 1.2. A physical experiment showing the fractal nature of the wet region (dark area).

We conclude from the above that it is necessary to take into account the fluctuations of $k(x)$ in order to get a good mathematical description of the flow. But how can we take these fluctuations into account when we do not know exactly what they are?

We propose the following: The lack of information about $k(x)$ makes it natural to represent $k(x)$ as a stochastic quantity. From a mathematical point of view, it is irrelevant if the uncertainty about $k(x)$ (or some other parameter) comes from "real" randomness (whatever that is) or just from our lack of information about a non-random, but complicated, quantity. If we accept this, then the right mathematical model for such situations would be partial differential equations involving stochastic or "noisy" parameters - stochastic partial differential equations (SPDEs) for short. The fundamental concepts of stochastic differential equations theory are $white\, noise\, W_t(t)$, the Ito integral $\int_0^t \phi(s,\omega)dB_s(\omega)$ and the Stratonovich integral $\int_0^t f(s,\omega) \circ dB_s$, where $B_s = B_s(\omega)$, $s \geq 0$ is n dimensional Brownian motion. Brownian motion was first noticed by Robert Brown in 1827, when he observed the presence of minute particles while examining the pollen grains of Clarkia pulchella suspended in water under a microscope. Since Brownian motion is not differentiable by definition, the integral state of SDE has been studied. The analytical solution of the SDE could not be found with normal analysis methods. Japanese mathematician Kiyoshi Itô made a great contribution to stochastic analysis by describing the "Itô lemma" in 1951. Analytical solutions of many SDEs can be found with this lemma. Nonlinear stochastic equations of evolution contribute a wide area in applications from chemistry to biology, physics, economics, and finance. So, finding exact solutions to the equations of nonlinear evolution is very significant for testing the accuracy of the numerical methods used, as well as the physical or mechanical problems that the equation models represent.

2. LITERATURE REVIEW

The lack of a specific method that can be used to reach exact solutions for all nonlinear equations has prompted researchers to develop many methods to find exact solutions to nonlinear equations of evolution, in which the search for solutions to study nonlinear physical phenomena plays a significant role.

In recent years, researchers have developed many powerful methods and used in solving nonlinear evolution equations to obtain wave solutions. In the following we review some of these methods, to name but a few; Hirota's bilinear transformation (Hirota, 1971; J. Yan, 2011), inverse scattering (Ablowitz and Clarkson, 1991), Weierstrass elliptic function (Chen and Wang, 2005; Kudryashov, 1990), Cole-Hopf transformation (Salas and Gómez S., 2010), (G'/G)-expansion (Wang et al., 2008), (1/G')-expansion (Yokucs and Durur, 2019; Yokus and Kaya, 2017), generalized Riccati equation (Z. Yan and Zhang, 2001), truncated Painleve expansion (Weiss et al., 1983; S.-L. Zhang et al., 2002), Backlund transformation (Lu, 2012; Singh and Gupta, 2016), F-expansion (Abdou, 2007; Wang and Li, 2005), homogeneous balance (Wang et al., 1996), Jacobi elliptic function expansion (Liu et al., 2001; Z. Yan, 2003), tanh-coth (Abdel-All et al., 2011; Fan, 2000; Malfliet, 1992; Wazwaz, 2007), direct algebraic (Soliman and Abdo, 2012), exp-function (He and Wu, 2006; Mohyud-Din et al., 2010; Naher et al., 2011, 2012), multi-wave (Shi et al., 2010) methods and so on. Different exact solutions (such as periodic wave, shock wave, solitary wave solutions etc.) are obtained using the above-mentioned methods. Due to the stochastic terms, it contains, the stochastic equations of evolution are more difficult to analyze than the deterministic equations of evolution, and there are not many studies on this topic. In modeling of turbulence of dispersive shallow water wave and in nonlinear wave propagation in noisy plasmas, the stochastic KdV equation come to light (Conte, 2003; Herman, 1990). In the scope of this thesis, some KdV-type equations are treated. The applicability of the above methods will be tested to search for exact solutions to the stochastic evolution of nonlinear equations given below.

1. *The stochastic Korteweg–de Vries –Burgers (KdV-B) equation*

The KdV-B equation can be written as

$$u_t + uu_x - vu_{xx} + \mu u_{xxx} = 0, \tag{2.0.1}$$

where $\mu \in R$ is the dispersion and $v \geq 0$ is the dissipation coefficient. The equation (2.0.1) is used in physics and other fields in modeling wave processes in dissipative-dispersive systems (N.A. Kudryashov, 1991). Fu and Liu (2011) studeid the existence of travelling wavefronts of the KdV–Burgers equation from a monotone dynamical systems point of view and obtained a sufficient condition for the existence. The stochastic form of Equation (2.0.1) can be written as

$$u_t + uu_x - vu_{xx} + \mu u_{xxx} = \eta(t), \tag{2.0.2}$$

by adding the $\eta(t)$ noise term. Richards (2014) has investigated local well posedness of stochastic KdV-Burgers equation with cylindrical white noise. The results obtained in (Richards, 2014) are given in the following theorems.

Theorem 2.1 (Local Well-posedness). Given $0 < \varepsilon < \frac{1}{16}$*, let* $s \geq -\frac{1}{2} - \varepsilon$*. Suppose* $\phi \in HS(L^2; H^{s+1-2\varepsilon})$ *is a Hilbert–Schmidt operator from* $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ *in the form*

$$\widehat{(\phi f)}(n) = \phi_n \hat{f}(n). \tag{2.0.3}$$

Then the following stochastic Korteweg–de Vries (KdV)–Burgers equation

$$\begin{cases} du = (u_{xx} - u_{xxx} - (u^2)_x)dt + \phi \partial_x dW, t \geq 0, x \in \mathbb{T} \\ u|_{t=0} = u_0, \end{cases} \tag{2.0.4}$$

is locally well posed in $H^s(T)$ *for mean zero data. That is, if* $u_0 \in H^s(T)$ *has mean zero, there exists a stopping time* $T_\omega > 0$ *and a unique process* $u \in C\big([0, T_\omega]; H^s(T)\big)$ *satisfying* (2.0.4) *on* $[0, T_\omega]$ *a.s.*

Theorem 2.2 (Global Well-posedness). Let $\phi \in HS(L^2; H^1)$*. Then* (2.0.4) *is* GW *in* $L^2(T)$ *for mean zero data. That is, if* $u_0 \in L^2(T)$ *has mean zero, then for any* $T > 0$ *there is a unique process* $u \in C\big([0, T]; L^2(T)\big)$ *satisfying* (2.0.4) *on* $[0, T]$ *a.s.*

2. *The stochastic Korteweg–de Vries (KdV) equation*

The celebrated KdV equation known as a simple nonlinear dispersive wave equation appearing in the literature is one of the simplest and most useful nonlinear model equations for solitary waves, and it represents the longtime evolution of wave phenomena. If the shape of the wave changes over time due to the wavelength or frequency of the wave speed, such waves are called dispersive waves (Zabusky and Kruskal, 1965). KdV equation is in the form;

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2.0.5}$$

The waves that are the solution of the KdV equation are called solitons. Existence, uniqueness, and continuous dependence on the initial data are proved for the local solution of the (generalized) Korteweg-de Vries equation was studied in (Ginibre et al., 1990; Kato, 1979). The stochastic form of equation (2.0.5) can be written as;

$$u_t + 6uu_x + u_{xxx} = \eta(t). \tag{2.0.6}$$

Wadati in (Wadati, 1983; Wadati and Akutsu, 1984) first analyzed the equation of stochastic KdV using analytic way and then he set the long -time behavior under Gaussian noise for single soliton solutions. In addition, several authors have studied stochastic KdV equation, for example (de Bouard et al., 1999; de Bouard and Debussche, 1998; Debussche and Printems, 1999; Konotop and Vázquez, 1994; Printems, 1999), and so on.

Theorem 2.3. (de Bouard and Debussche, 1998) *Assume that* $u_0 \in L^2\big(\Omega; H^1(R)\big) \cap L^4\big(\Omega; L^2(R)\big)$ *and is* $\mathcal{G}_0$ *-measurable, and* $\Phi \in L_2^0\big(L^2(\mathbb{R}); H^1(\mathbb{R})\big)$*; then there exists a unique solution of*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\left(u\frac{\partial u}{\partial x}\right)d\,s + \int_0^t S(t-s)\Phi d\,W(s), \tag{2.0.7}$$

in $X_\sigma(T_0)$ *almost surely, for any* $T_0 > 0$ *and for any* σ *with* $3\,/\,4 < \sigma < 1$ *.Moreover,* $u \in L^2\left(\Omega; L^\infty\big(0, T_0; H^1(R)\big)\right)$.

Theorem 2.4. (de Bouard and Debussche, 1998) *Assume that* $\Phi \in L_2^0\left(L^2(R), H^{\tilde{\sigma}}(R)\right)$ *for some* $\tilde{\sigma} > 3/4$.

Then $\bar{u}$ *is almost surely in* $X_\sigma(T)$ *for any* $T > 0$ *and any* σ *such that* $3/4 < \sigma < \tilde{\sigma}$. *Moreover*

$$E\left(|\bar{u}|^2_{X_\sigma(T)}\right) \leq C(\sigma, \tilde{\sigma}, T)|\Phi|^2_{L_2^{0,\tilde{\sigma}}}.$$

3. *The stochastic Burgers' equation*

Expressed as

$$u_t + uu_x + u_x = 0, \tag{2.0.8}$$

Burgers equation is one of the most important nonlinear propagation equations. This equation is the simplest nonlinear equation model for propagating waves in fluid dynamics. It was first used by Burger to describe one-dimensional turbulence (Burgers, 1995). Benia and Sadallah (2016) established the existence, uniqueness and the optimal regularity of the solution in the anisotropic Sobolev space for Burgers equation. The stochastic form of equation (2.0.8) can be written as

$$u_t + uu_x + u_x = \eta(t). \tag{2.0.9}$$

Here the term $\eta(t)$ refers to external noise. There are several existence and uniqueness results in the literature for mild solutions of stochastic Burgers equations driven by colored noise (Giuseppe Da Prato and Gatarek, 1995).

Theorem 2.5. (Giuseppe Da Prato and Gatarek, 1995) *Let u_0 be given which is $\mathcal{F}_0$ - measurable and such that $u_0 \in L^2(0,1)$*
a.s. and let $T > 0$. Then there exists a unique mild solution of equation

$$\begin{cases} du = \left(Au + \dfrac{1}{2}\dfrac{\partial}{\partial x}(u^2)\right)dt + g(u)dW \\ u(0) = u_0. \end{cases} \tag{2.0.10}$$

Theorem 2.6. (Giuseppe Da Prato and Gatarek, 1995) *There exists a unique invariant measure for the equation*

$$\begin{cases} du = \left(Au + \dfrac{1}{2}\dfrac{\partial}{\partial x}(u^2)\right)dt + g(u)dW \\ u(0) = u_0. \end{cases} \tag{2.0.11}$$

and by space-time white noise (Guiseppe Da Prato et al., 1994) in the case of cylindrical Wiener process.

4. *The stochastic Kuramoto - Sivashinsky (KS) equation*

Kuramoto-Sivashinsky equation

$$u_t + uu_x + u_{xx} - \nu u_{xxxx} = 0, \tag{2.0.13}$$

arises in a wide variety of phenomena such as reaction-diffusion systems, two-phase flows in cylindrical geometries, flame propagation and viscous film flow. Eq. (2.0.1) was introduced independently by Kuramoto's study (Kuramoto and Tsuzuki, 1975) on the analysis of reaction-diffusion systems and by Sivashinsky's study (Sivashinsky, 1977) on instability in laminar flame and fluid dynamics. Although Eq. (2.0.1) is one of the simplest equations, it models chaotic behavior of complicated dynamics. As the parameter ν decreases, the large time behavior of the system changes from steady-state solutions to chaotic solutions. Kudryashov (2013) studied dissipative KS equation. He showed that the dissipative KS equation does not possess solitary wave solutions and has only rational solutions (quasi-exact solutions). Stochastic nonlinear Kuramoto-Sivashinsky equation can be given as

$$u_t + uu_x + u_{xx} - u_{xxxx} = \eta(t). \tag{2.0.14}$$

Stochastic KS equation was studied in many papers from different perspectives. Duan and Ervin (2001) studied the existence and uniqueness of solutions of the stochastic KS equation, the results of which are given below.

Theorem 2.9. (Local Existence and Uniqueness) For u_0 in H there exists a random variable τ taking values $P - a.s.$ in $(0, T]$ such that problem

$$\begin{gathered} du + (u_{xxxx} + u_{xx} + uu_x)dt - dw = 0 \\ u(0, x) = u_0(x), -l < x < l, \textit{ and } u(t, -l) = u(t, l) = 0 \textit{ for } t > 0, \end{gathered} \tag{2.0.15}$$

has a unique solution u on the interval $[0, \tau]$.

*Theorem 2.10. (Global Existence) For $u_0 \in H = L^2(I)$, there exists P a.s. a unique solution $u(\cdot, x) \in E$ of equations (*2.0.1*).*

Yang (2006) obtained a pull-back random attractor for the initial-boundary value problem of the stochastic KS equation. Existence and uniqueness of invariant measures for the stochastic KS equation was investigated in (Ferrario, 2008). (Wu et al.(2018) provided sufficient conditions guaranteeing global well-posedness of the stochastic generalized KS equation with a multiplicative noise. Gao et al.(2018)

simulated the effect of different noises on the solitary wave solutions of the stochastic KS equation by using finite difference method.

5. *The stochastic Kawahara equation (KH) equation*

The Kawahara equation has formed as following:

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0, \tag{2.0.16}$$

is a dispersive fifth-order equation arising in the modeling of magneto-acoustic waves in a plasma and small-amplitude water waves with surface tension and was introduced by Kawahara (Kawahara, 1972). The equation is also known as a special case of the Benney-Lin equation, singularly perturbed KdV equation or fifth-order KdV equation. Many studies have been carried out for solutions to the deterministic Kawahara equation. Kabakouala and Molinet (2018) studied orbital stability of solitary waves of a generalized Kawahara equation in $H^2(R)$ by using spectral method. Kwak (2020) is concerned with global well posedness of a periodic modified Kawahara equation in $L^2(T)$. Mancas (2019) obtained travelling wave solutions of Kawahara equation by using elliptic function method. Biswas (2009) found travelling wave solutions of a generalized Kawahara equation. The existence of compaction solutions and solitary patterns solutions for Kawahara type equation is demonstrated in (Wazwaz, 2003). By adding the external noise term $\eta(t)$, the nonlinear stochastic evolution equation can be given as:

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = \eta(t). \tag{2.0.17}$$

Some studies were performed on stochastic Kawahara equation. (Hyder and Zakarya (2019) studied local well-posedness of solutions of a modified Kawahara equation in $H^s(R), s > \frac{-7}{4}$ and global well-posedness in $L^2(R)$ by using the fixed-point argument and the Fourier restriction method.

Theorem 2.11. Assume that $s \succ \frac{7}{4}, \Phi \in L_2^{0,s}, b \in \left(0, \frac{1}{2}\right)$ *and* b *is close enough to* $\frac{1}{2}$*. If* $u_0 \in H^s(R)$ *for almost surely* $\omega \in \Omega$ *and* u_0 *is* $\mathcal{F}_0$ *−measurable. Then for almost surely* $\omega \in \Omega$*, there exists a constant* $T_\omega > 0$ *and a unique solution* u *of the Cauchy problem*

$$\begin{cases} du + (\alpha u_{5x} + \beta u_{3x} + \gamma u_x + \mu u u_x)dt = \Phi dW(t) \\ u(x,0) = u_0(x), \end{cases} \qquad (2.0.18)$$

on $[0, T_\omega]$ *which satisfies:*

$$u \in L^2\left(\Omega; C\left([0, T_0]; H^s(R)\right)\right), \quad \text{for any} \, T_0 > 0$$

Therefore, in the case of $s = 0$, *we can obtain a global existence result for*

$$u_t + \alpha u_{5x} + \beta u_{3x} + \gamma u_x + \mu u u_x = \Phi \frac{\partial^2 B}{\partial t \partial x} \qquad (2.0.19)$$

where $\alpha \neq 0, \beta$, *and* γ *are real numbers;* μ *is a complex number;* u *is a stochastic process defined on* $(x, t) \in R \times R_+$; *is a linear operator; and* B *is a two-parameter Brownian motion on* $R \times R_+$.

Precisely, the following theorem was stated for global existence:

Theorem 2.12. Let $u_0 \in L^2(\Omega, L^2(R))$ *be an* $\mathcal{F}_0 -$ *measurable initial data, and let* $\Phi \in L_2^{0,0}$ *Then, the solution u given by Theorem 2.1 is global and satisfies:*

$$u \in L^2\left(\Omega; C\left([0, T_0]; H^s(R)\right)\right), \quad \text{for any } T_0 > 0.$$

Agarwal et al.(2020) constructed local and global well-posedness of a modified stochastic Kawahara equation by using the same arguments of (Hyder and Zakarya, 2019).

6. *The Wick-type stochastic extended KdV equation*

The name extended KdV equation was given by Bakırtaş and Antar in (Bakırtaş and Antar, 2003), and Bakırtaş and Demiray (Bakırtaş and Demiray, 2005) to the following equation

$$u_t + \nu_1 u u_x + \nu_2 u_{xxx} + \mu(t) u_x = 0, \qquad (2.0.20)$$

where ν_1 and ν_2 are constants due to the initial deformation of the tube material, and $\mu(t)u_x$ represents the contribution of the tapering of tube. They studied the weakly nonlinear propagation of waves in elastic tubes filled with non-compressible viscous fluid by using the method of reductive perturbation. They obtained the equation by treating blood as a non-compressible viscous fluid and the arteries as tapered, flexible, thin-walled, long circular conical tube. Also (Karczewska and Szczecinski (2019) studied the existence and uniqueness of the mild solution of the stochastic extended KdV equation in the following form

$$du + \left(\frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} + u\frac{\partial^3 u}{\partial x^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right)dt = \Phi dW, \tag{2.0.21}$$

where W is a cylindrical Wiener process. They obtained sufficient conditions for the existence and uniqueness of a local mild solution using a fixed point argument as follows.

Theorem 2.13. Assume that $u_0 \in_2 L^2\big(\Omega; H^1(R)\big) \cap_2 L^4\big(\Omega; L^2(R)\big)$ *and it is* $\mathcal{F}_0$ *-measurable and* $\Phi \in L_2^0\big(L^2(R); H^1(R)\big)$. *If* $\frac{\partial^2}{\partial x^2}\left[\int_0^t V(t-s)\Phi dW(s)\right] \in L^2\big(\Omega; L_x^2(L_t^\infty)\big)$ *holds then there exists a unique mild solution to the equation (*2.0.2*) with initial condition* $u(x,0) = u_0(x)$, $x \in R$, $t \geq 0$, *such that* $u \in_2 X_\sigma(T)$ *almost surely for some* $T > 0$ *and for any* $\sigma \in \left(\frac{3}{4}, 1\right)$.

The Wick-type stochastic extended KdV-equation for equation (2.0.2) is given in the following form

$$U_t + H_1(t) \diamond U_x + H_2(t) \diamond U \diamond U_x + H_3(t) \diamond U_{xxx} = 0, \tag{2.0.22}$$

where $\diamond$ is the Wick product on the Hida distribution space$\big(S(R^d)\big)^*$ and $H_i(i = 1,2,3)$ are the white noise functions. More recently, the white noise function approach has been applied to the stochastic partial differential equations by several authors and obtained many new solutions to the stochastic partial differential equation. Xie et al in (Xie, 2003) found stochastic soliton solution using Hermit transformation and homogeneous balance method for Wick-type stochastic KdV equation. In (Chen and Xie, 2005), Chen et al obtained a series of soliton-like solutions to the Wick-type stochastic KdV equation with the help of Hermit transformation and an algebraic method. In addition, some solutions of the Jacobian elliptic function for two types of stochastic KdV equation were derived by Wei et al in (Wei et al., 2005) by Hermit transformation and the method of Jacobi elliptic function expansion.

In this thesis study, some analytical solution methods will be employed to obtain exact solutions of nonlinear stochastic differential equations. Analytical solutions of nonlinear stochastic differential equation are obtained using the combination of Galilean transformation and nonlinear methods. Moreover, a Wick-type stochastic extended KdV equation is investigated, and the solutions will be interpreted with graphs. More clearly, the answer to the following questions

- How are solutions of nonlinear evolution equations influenced by external noise?
- Can tanh and extended tanh be used in order to find exact solutions to stochastic evolution equations?
- Can F-expansion method and Hermit transform be used for the nonlinear Wick-type stochastic extended KdV with Gaussian white noise to obtain exact elliptic function solutions?

will be investigated.

3. MATERIALS AND METHODS

3.1. Preliminaries

In this section, we provide some basic definitions, theorems, and the outlines of the methods we used throughout the thesis. Since the solutions of stochastic differential equations are stochastic processes defined in a probability space, we should firstly give basic definitions on probability theory for a better understanding of the thesis.

3.1.1. Probability theory

Probability theory deals with mathematical models, the results of which are linked to chance. The probability function, which is given in the following form:

$$P: \{\text{Set of Events}\} \to [0,1],$$

has a history that goes back to the 17th century (Cáceres, 2017). The above definition may be interpreted as probability function is ascribing real numbers in $[0,1]$ to subsets of $A \subseteq \Omega$, where Ω is the possible outcomes of an event, which is called sample space. The structure of the probability function was shaped by the axioms put forward by the Russian mathematician A. N. Kolmogorov in 1933.

Definition 3.1. σ-Algebra (Grimmet and Stirzaker, 2020). A collection of subsets F of Ω is called a σ-algebra if the following properties are satisfied:

1. $\phi \in F$ and $\Omega \in F$;
2. $A \in F \Rightarrow A^c \in F$ (F is closed under complements);
3. $A_i \in F$ for $i = 1,2,3,\ldots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$ (F is closed under countable unions).

The elements of F are named measurable sets, and the pair (Ω, F) is named a measurable space.

Definition 3.2. Probability Measure (Grimmet and Stirzaker, 2020). A measure μ on (Ω, F) is a probability measure, if it fulfills the following conditions:

1. $0 \le \mu(A) \le 1$ for $A \in F$.
2. $\mu(\phi) = 0$ and $\mu(A) = 1$.
3. If $\{\omega_i\}_{i=1}^{\infty} \subseteq F$, then $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

Definition 3.3. Probability Space (Koralov and Sinai, 2007).The triplet (Ω, F, μ) is called a probability space, if the measure of entire sample space is equal to one: $\mu(\Omega) = 1$, where (Ω, F) is a measurable space. Hereafter, for a probability space, the measure μ will be denoted by P for convenience.

Definition 3.4. Filtration (Koralov and Sinai, 2007). Let (Ω, F, μ) be a probability space. A collection of increasing sub-sigma algebras of F is called a filtration. In other words, if $F_t \subseteq F$, $t \in T$ then for all $s \leq t$, $F_s \subseteq F_t$, where T is a parameter set.

Definition 3.5. Stochastic Process (Guiseppe Da Prato and Zabczyk, 2014). A family of random variables $\{X(t)\}_t \in I$ defined on some probability space (Ω, F, P) is called a stochastic process, where I is an interval of R^1.

3.1.2. Stochastic calculus

The foundations of stochastic processes were laid when Robert Brown noticed the erratic movements of pollen particles floating on the water. The physical interpretation of the movement is explained by Einstein in 1905, by random collisions with water molecules. The complete mathematical treatment is provided by Wiener in 1923. This mathematical treatment by Wiener still plays an important role in the theory of stochastic processes. The Wiener integral constructed by Wiener was later developed by Ito in 1942 which is known as the Ito integral. In this subsection, we mention some basic concepts such as Brownian motion, white noise, stochastic integrals, Ito's formula which are basis for the other sections.

Definition 3.6. Brownian Motion. In 1827, the botanist Robert Brown detected the chaotic motion of pollen particles suspending in water but until the studies of Einstein, Perrin, and some other physicists a mathematical model was not provided. In 1905, Albert Einstein established a relation between the microscopic random motion of particles and the macroscopic diffusion equation confirming the existence of atoms. In the following, we will summarize this connection (Sarkka, 2012).

Take n particles floating in a liquid and suppose that Δt is a small-time interval. Let us denote the displacement in x −coordinates of particles by Δ over the time interval Δt. The number of particles displacing between Δ and $\Delta + \mathrm{d}\Delta$ is given by

$$\mathrm{d}n = n\psi(\Delta)\mathrm{d}\Delta,$$

where $\psi(.)$ is a symmetric probability density satisfying $\psi(x) = \psi(-x)$ and is nonzero only for very small values of Δ. Denoting the number of particles per unit volume by $w(x,t)$, the number of particles located at $x + \mathrm{d}x$ in time $t + \Delta t$ may be derived from $w(x,t)$ as

$$w(x, t + \Delta t) = \int_{-\infty}^{\infty} w(x + \Delta, t)\psi(\Delta)\mathrm{d}\,\Delta. \tag{3.1.1}$$

Since Δt assumed to take very small values, one can write

$$w(x, t + \Delta t) = w(x,t) + \Delta t \frac{\partial w(x,t)}{\partial t}, \tag{3.1.2}$$

and $w(x + \Delta, t)$ may be expressed in terms of Δ as a Taylor series as follows

$$w(x + \Delta, t) = w(x,t) + \Delta \frac{\partial w(x,t)}{\partial x} + \frac{\Delta^2}{2}\frac{\partial^2 w(x,t)}{\partial x^2} + \cdots \tag{3.1.3}$$

Using (3.1.2) and (3.1.3) in (3.1.1) yields

$$w(x,t) + \Delta t \frac{\partial w(x,t)}{\partial t} = w(x,t)\int_{-\infty}^{\infty}\psi(\Delta)\mathrm{d}\,\Delta + \frac{\partial w(x,t)}{\partial x}\int_{-\infty}^{\infty}\Delta\psi(\Delta)\mathrm{d}\,\Delta + \frac{\partial^2 w(x,t)}{\partial x^2}\int_{-\infty}^{\infty}\frac{\Delta^2}{2}\psi(\Delta)\mathrm{d}\Delta + \cdots$$

In the above all the odd order terms vanish. Consideration of $\int_{-\infty}^{\infty}\psi(\Delta)\mathrm{d}\,\Delta = 1$, and setting

$$\int_{-\infty}^{\infty}\frac{\Delta^2}{2}\psi(\Delta)\mathrm{d}\Delta = D,$$

yields the diffusion equation

$$\frac{\partial w(x,t)}{\partial t} = D\frac{\partial^2 w(x,t)}{\partial x^2}.$$

A relation was derived by Einstein for \$D\$ in terms of atomic properties of the matter

$$D = \frac{RT}{N6\pi\eta r},$$

where η is the viscosity of liquid, T is the temperature, r is the radius of the Brownian particles, N is the Avogadro's number, and R is the gas constant. Using the above diffusion constant, mean squared displacement of the particles were predicted by Einstein as

$$z(t) = \frac{RT}{3N\pi\eta a}t,$$

where a is the diameter of the particles.

Although Einstein presented a consistent explanation to the Brownian motion, Langevin's equation

$$d\boldsymbol{u}/dt = -\beta\boldsymbol{u} + \boldsymbol{A}(t),$$

containing random and frictional force terms is a milestone in the modern theory of the Brownian motion of a free particle. Here $\boldsymbol{u}$ is the velocity of the particle. With respect to the above equation, the effect of the external environment on the motion of the particle can be divided into two parts: first, the effect of dynamical friction experienced by the particle, $-\beta\boldsymbol{u}$, and second, the main characteristic of the Brownian motion, $\boldsymbol{A}(t)$, the fluctuating part (Chandrasekhar, 1943). The illustrations of Einstein's and Langevin's model are given in the following figure.

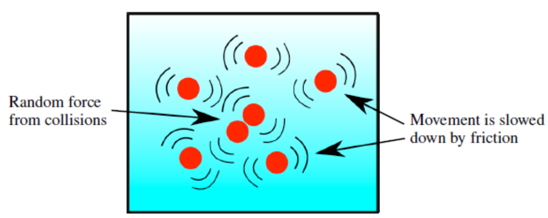

Figure 3.1. Illustration of Langevin's model of Brownian motion.

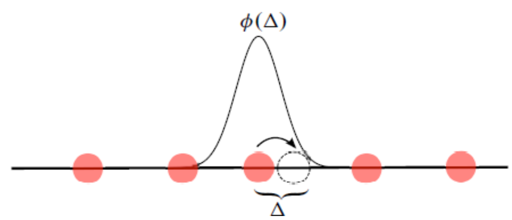

Figure 3.2. Illustration of Einstein's model of Brownian motion.

Now, we shall define one-dimensional Brownian motion on a probability space.

Definition 3.7. ***Standard Brownian Motion*** (Koralov and Sinai, 2007). A system of random processes $\{B(t), t \in [0,\infty)\}$ on a probability space (Ω, F, P) is called a *Brownian motion* if

1. $B(0) = 0$.
2. For all $0 \le t_1 < t_2$, $B(t_2) - B(t_1) \sim N(0, t_2 - t_1)$.
3. $E\left[\left(B(t) - B(s)\right)^2\right] = |t - s|$.
4. $B(t)$ has continuous sample paths.

The covariance function is defined by $\Gamma(s,t) = E[B(s)B(t)] = \frac{1}{2}(|s| + |t| - |s - t|)$.

Definition 3.8. White Noise. (Sarkka, 2012). A real-valued Gaussian process $W(t)$ with the following properties is called a white noise:

1. If $t_1 \neq t_2$, then $W(t_1)$ and $W(t_2)$ are independent.
2. For the map $t \longmapsto W(t)$,

$$m_\omega(t) = E[W(t)] = 0,$$

$$C_\omega(t,s) = E[W(t)W^T(t)] = \delta(t-s)Q,$$

where δ is the Dirac-delta and Q is the spectral density of the process.

From the properties of the white noise, one can also conclude that the sample path of $t \mapsto W(t)$ is discontinuous a.e. The following figure illustrates a scalar white noise process.

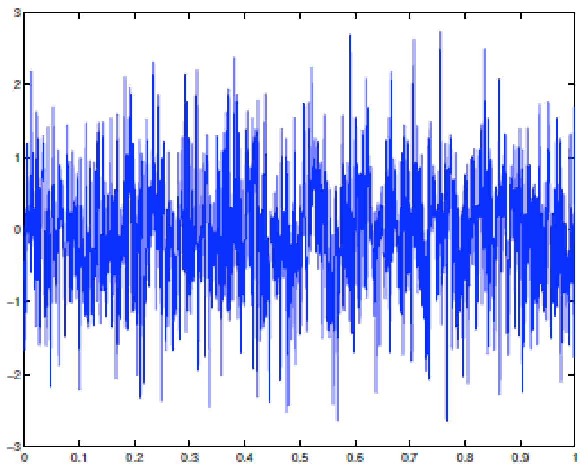

Figure 3.3. White noise.

For most of the applications, the noise function does not satisfy the first property, i.e., when t_1 and t_2 are close enough $W(t_1)$ and $W(t_2)$ are not independent. The notion of smoothed white noise is adapted for this case as given in the following way:

$$W_\phi := W_\phi(t,\omega) := \langle \omega, \phi_t \rangle = \int \phi_t(s) d\, B_s(\omega),$$

where t is the time, ω is a random element, the integral is Ito integral which will be defined below, ϕ is a test function, and ϕ_t is the t-shift of ϕ given as $\phi_t(s) = \phi(s-t)$, $s, t \in R$.

There is a connection between white noise and Brownian motion. We can roughly say that white noise is formal derivative of Brownian motion. Integration by parts for Wiener-Itô integrals yields

$$\int_{\boldsymbol{R^d}} \phi(x) d\,B(x) = (-1)^d \int_{\boldsymbol{R^d}} \frac{\partial^d \phi}{\partial x_1 \dots \partial x_d}(x) B(x) d\,x.$$

Accordingly

$$W(\phi) = \int_{R^4} \phi(x) d\,B(x) = \left((-1)^d \frac{\partial^i \phi}{\partial x_1 \dots \partial x_d}, B\right) = \left(\phi, \frac{\partial^4 B}{\partial x_1 \dots \partial x_d}\right),$$

where $(.,.)$ is the usual inner product in the space $L^2(R^d)$. Namely, in the sense of distributions we have

$$W = \frac{\partial^d B}{\partial x_1 \dots \partial x_d}.$$

Theorem 3.1. (*Ito's formula*). *For any function $f(t,x)$; $f \in C^{1,2}$ that has two continuous derivatives with respect to x and continuous derivative with respect to t. Then, the process $f\big(t, X(t)\big)$ satisfies*

$$df\big(t, X(t)\big) = \left[f_t + \frac{1}{2}\sigma^2 f_{xx}\right]\big(t, X(t)\big)dt + f_x\big(t, X(t)\big)dX(t), \qquad (3.1.5)$$

where

$$dX(t) = \mu\big(t, X(t)\big)dt + \sigma\big(t, X(t)\big)dW(t), \qquad (3.1.6)$$

*is a diffusion process. Now by substituting (*3.1.6*) in (**Error! Reference source not found.**) w e can obtain the useful form of the following Ito rule:*

$$df\big(t, X(t)\big) = \left[f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx}\right]\big(t, X(t)\big)dt + (\sigma f_x)\big(t, X(t)\big)dW(t). \qquad (3.1.7)$$

Definition 3.9. Box Algebra (Steele, 2001). This is an algebra for linear combination of the formal symbols dt and dB_t In this algebra, the addition operation is usual algebraic addition, and their products are found by traditional rules of transitivity and associativity Please (see Table 3.1, Table 3.2).

Table 3.1. Box Algebra Multiplication Table

$\cdot$	$d\,t$	dB_t
$d\,t$	0	0
dB_t	0	$d\,t$

Table 3.2. Extended Box Algebra

$\cdot$	$d\,t$	dB_t^1	dB_t^2
$d\,t$	0	0	0
dB_t^1	0	$d\,t$	0
dB_t^2	0	0	$d\,t$

The experience with Itô's formula as a tool for understanding the dB_t integrals now leaves one with a natural question: Is there an appropriate analog of Itô's formula for dX_t integrals? That is, if the process X_t can be written as a stochastic integral of the form

$$\int_0^t f(\omega,s)d\,X_s \overset{\text{def}}{=} \int_0^t f(\omega,s)a(\omega,s)d\,s + \int_0^t f(\omega,s)b(\omega,s)d\,B_s, \tag{3.1.8}$$

and if $g(t,y)$ is a smooth function, can we then write the process $Y_t = g(t,X_t)$ as a sum of terms which includes a dX_t integral?

Of course, there is a positive answer to these questions, and it turns out that it is well expressed with a simple formal aid usually called the box calculus, though the term box algebra would be more precise. This is an algebra for the set $\mathcal{A}$ of linear combinations of the formal symbols $d\,t$ and dB_t where adapted functions are regarded as the scalars. In this algebra, the addition operation is just the usual algebraic addition, and products are then computed by the traditional rules of associativity and transitivity together with a multiplication table for the special symbols $d\,t$ and dB_t. The new rules one uses are simply

$$dt\cdot dt = 0, \quad dt\cdot dB_t = 0, \quad \text{and} \quad dB_t\cdot dB_t = dt.$$

As example for the application these rules were apply, it can be checked the product

$$(adt + bdB_t)\cdot(\alpha dt + \beta dB_t),$$

can be simplified by associativity and commutativity to give

$$a\alpha dt\cdot dt + a\beta dt\cdot dB_t + b\alpha dB_t\cdot dt + b\beta dB_t\cdot dB_t = b\beta dt.$$

If one uses this formal algebra for the process X_t which we specified in longhand by using

$$X_t = \int_0^t a(\omega,s)d\,s + \int_0^t b(\omega,s)d\,B_s, \tag{3.1.9}$$

or in shorthand by using

$$dX_t = a(\omega, t)dt + b(\omega, t)dB_t, \quad X_0 = 0. \tag{3.1.10}$$

Hence one can arrive at the following version of general formula Itô:

$$df(t, X_t) = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2} f_{xx}(t, X_t)dX_t \cdot dX_t. \tag{3.1.11}$$

This simple formula sums up an enormously useful amount of valuable information as well as being exceptionally productive.

The proof of $dW_t . dt = 0$. Someone always ask why dw times dt is zero or why dt squared is zero.

$$dW_t dt = 0, \tag{3.1.12}$$

the answer may vary but it can be as simple as saying if dt is 0.1 then dt squared it will be 0.01. We will provide more convincing proof of this below. First, let's write the equation (3.1.12) in the following integral form

$$\int_0^t d\, W_s ds, \tag{3.1.13}$$

now, applying what we have learned in deterministic calculus, we know that the integral is the limit of discrete sum within the interval form 0 to t. Let's now take the interval from 0 to t and divide it into n of length Δt as in (Figure 3.4).

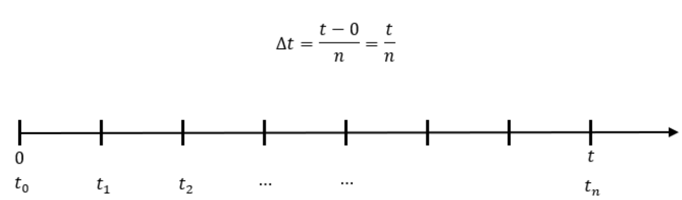

Figure 3.4. Split interval from 0 to t

Now we split the interval and let us represent the endpoints of the intervals by t_i with i going from 0 to n the number of sub intervals and one can then approximate the integral via sum of ΔW times Δt over these sub intervals hoping that as we increase the number of sub intervals the approximation will give the value of the integral.

$$\int_0^t d\,W_s ds = \lim_{n\to\infty}\sum_{k=1}^{n}\Delta W_{t_k}\Delta t_k = \lim_{n\to\infty}\sum_{k=1}^{n}\left(W_{t_k}-W_{t_{k-1}}\right)(t_k-t_{k-1})\,, \tag{3.1.14}$$

By the way ΔW and Δt are just the changes in the value of W and t over the sub intervals so easy to calculate once one has a Brownian path over time. Let's now substitute ΔW times Δt by $x_k = \Delta W_{t_k}\Delta t_k$ to convert the thing into something familiar and let us represent the sequence by $X_n = \sum_{k=1}^{n} x_k$. Now in the calculation, we are quite accustomed to limiting subsequent iterations and let's say the limiting value is X nice and easy.

$$\int_0^t d\,W_s ds = \lim_{n\to\infty} X_n = X, \tag{3.1.15}$$

but this limit won`t do here reason being our axes are random variables. So, this some probability associated with them but the limit we have doesn't contain any probability and stochastic calculus. One talks about convergence and there are several modes of such conversions some easier to prove than others. The most common one used in the stochastic integration problem we have here is mean square conversion. So, mean square convergence is what we're going to use we say the sequence X_n converges to X in the mean square if the expected value of the squared deviation from X goes to zero as n become very large $\lim\limits_{n\to\infty} E\,[|X_n - X|^2] = 0$. we are claiming that our sequence converges to zero. So, we replace X by zero so $\lim\limits_{n\to\infty} E\,[|X_n - 0|^2] = 0$ and now we can replace X_n by the sum then we get $\lim\limits_{n\to\infty} E\,[(\sum_{k=1}^{n} x_k)^2] = 0$ and we are done with the capital X. We only introduce this symbol so that we can see all we are doing is taking the limit of the sequence in some sense

$$\int_0^t d\,W_s ds = \lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n} x_k\right)^2\right] = 0. \tag{3.1.16}$$

Now we can split the square of the sum and square in class terms so this is just a multivariate version of $(a+b)^2 = a^2 + b^2 + 2ab$, so we get $(\sum_{k=1}^{n} x_k)^2 = \sum_{k=1}^{n} x_k^2 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1} x_k x_j$. Now let's make the substitution

$$\int_0^t d\,W_s ds = \lim_{n\to\infty} E\left[\sum_{k=1}^{n} x_k^2 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1} x_k x_j\right], \qquad (3.1.17)$$

we can replace $x_k = \Delta W_{t_k}\Delta t_k$

$$\int_0^t d\,W_s ds = \lim_{n\to\infty} E\left[\sum_{k=1}^{n} \Delta W_{t_k}^2 \Delta t_k^2 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1} \Delta W_{t_k}\Delta t_k \Delta W_{t_j}\Delta t_j\right], \qquad (3.1.18)$$

Expectation of the sum is the sum of expectations and we took Δt out of the expectation because it's deterministic

$$\int_0^t d\,W_s ds = \lim_{n\to\infty} \left(\sum_{k=1}^{n} E\left[\Delta W_{t_k}^2\right]\Delta t_k^2 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1} E\left[\Delta W_{t_k}\Delta W_{t_j}\right]\Delta t_k \Delta t_j\right), \qquad (3.1.19)$$

now we just need the properties of the Brownian increments. Brownian increments and non-overlapping intervals are independent so the second term $E\left[\Delta W_{t_k}\Delta W_{t_j}\right]$ is 0.

$$\int_0^t d\,W_s ds = \lim_{n\to\infty} \sum_{k=1}^{n} E\left[\Delta W_{t_k}^2\right]\Delta t_k^2 , \qquad (3.1.20)$$

then expected value of ΔW squared is equal to the length of the sub interval

$$\int_0^t d\,W_s ds = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta t_k \Delta t_k^2 , \qquad (3.1.21)$$

and we can combine the Δt term so we get Δt cube

$$\int_0^t d\,W_s ds = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta t_k^3 , \qquad (3.1.22)$$

now as n become very large this limit goes to 0 to see that let's consider the time interval from 0 to 1. Let us say we start with n equal to 1 as there is only one interval the length of the interval from 0 to 1 is obviously 1. Then we take Δt to the power 3 means we take the cube 1 which gives 1. Now let us increase n to 2 so the length of each sub interval is now 0.5 and we have two of them if we calculate the cube of Δt and sum across the intervals we get 0.25 so the sum of Δt to the power 3 has declined from 1 to 0.25 as we doubled end from 1 to 2. Now let's double and again to 4 so 4 sub intervals each are planned 0.25 and if we calculate the cube of Δt and each sub interval and sum across intervals we see it has declined again and now if we double the number of sub intervals to 8 so the sub interval is now upland 0.125 and if we calculate the cube of Δt and sum it we see the sum has declined if we continue increasing n we will see the sum of Δt to the power 3 will go to 0 and we just conclude that the integral of dW times dt goes to 0 mean square

$$\int_0^t d\,W_s ds = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta t_k^3 = 0 \Rightarrow dW_t dt = 0, \tag{3.1.23}$$

and this is what they mean when they say dW times dt is equal to zero (see Figure 3.5).

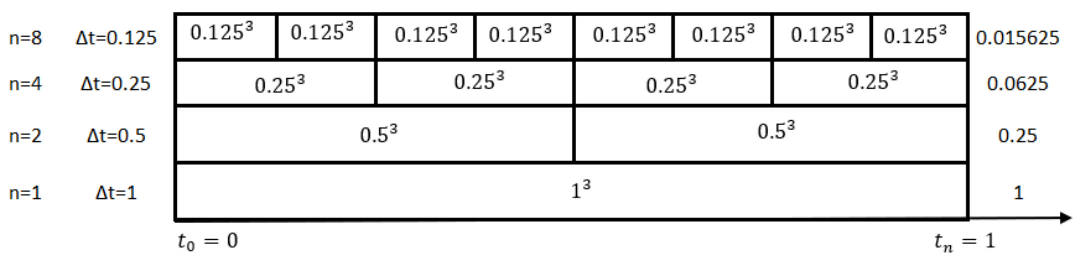

Figure 3.5. Calculating of the sum cube Δt

Now we can go and apply the same logic to dt squared which should be slightly simpler and dW squared which is slightly complicated.

The proof of $dt^2 = 0$. In the previous paragraph we explained why the term dW times dt is zero and now we use the same logic to find out why dt squared is equal to zero. It is going to be a bit easier because we do not have any stochastic term in dt squared. First as in the previous paragraph we can write dt^2 in the following form:

$$\int_0^t d\,s^2, \tag{3.1.24}$$

let us divide the interval from zero to t into n sub intervals, each of which is essentially Δt, we split the intervals into sub intervals along Δt and let us represent the endpoints of some periods on t_i with i going from 0 to n the number of sub intervals; (see Figure 3.4), so from (3.1.24) we obtain ;

$$\int_0^t d\,s^2 = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta t_k^2\,, \tag{3.1.25}$$

as we mentioned in the previous paragraph the most convergence used in the stochastic integration problem is mean square conversion, so let us see if the mean square convergence produces 0 for dt squared. We say the sequence X_n converges to X in the mean square if the expected value of the squared deviation from X goes to 0 as n become very large.

$$\lim_{n\to\infty} E\,[|X_n - X|^2] = 0, \tag{3.1.26}$$

now, in order to achieve our goal, we replace in the (3.1.26), X_n by the partial sum and X with zero

$$\lim_{n\to\infty} E\left[\left|\sum_{k=1}^{n} \Delta t_k^2 - 0\right|^2\right] = 0, \tag{3.1.27}$$

and we are claiming that the limit (3.1.27) goes to zero. Now in place of the point-wise limit (3.1.25) we write the probabilistic limit (3.1.27) to be consistent with the approach used for the other rules. So, we obtain.

$$\int_0^t d\,s^2 = \lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n} \Delta t_k^2\right)^2\right], \tag{3.1.28}$$

but the expected value upper deterministic term is the deterministic term and $\Delta t = \frac{t}{n}$, so we get;

$$\begin{aligned}\int_0^t ds^2 &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \Delta t_k^2\right)^2 \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \left(\frac{t}{n}\right)^2\right)^2 \\ &= \lim_{n\to\infty} \left(n\left(\frac{t}{n}\right)^2\right)^2 \\ &= \lim_{n\to\infty} \left(\frac{t^2}{n}\right)^2 = 0 \Rightarrow dt^2 = 0\end{aligned}, \qquad (3.1.29)$$

This is what they mean when they say dt^2 squared is equal to zero.

The proof of $dW^2 = dt$. By following the same steps in the previous two paragraphs, we can write;

$$\int_0^t d\,W_s^2 = \lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n} \Delta W_{t_k}^2 - t\right)^2\right], \qquad (3.1.30)$$

whereas we used the definition of the mean square convergence

$$\lim_{n\to\infty} E\,[|X_n - X|^2] = 0.$$

Then replaced X_n with $\sum_{k=1}^{n} \Delta W_{t_k}^2$ and t in place of X. We are claiming that the $\lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n} \Delta W_{t_k}^2 - t\right)^2\right] = 0$. After substituting each term in place, we get;

$$\begin{aligned}\int_0^t dW_s^2 &= \lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n} \Delta W_{t_k}^2 - t\right)^2\right] \\ &= \lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n} \Delta W_{l_k}^2\right)^2 + t^2 - 2t\sum_{k=1}^{n} \Delta W_k^2\right]\end{aligned}, \qquad (3.1.31)$$

now we know that;

$$(x_1 + x_2 + \cdots + x_n)^2 = \left(\sum_{k=1}^{n} x_k\right)^2 = \sum_{k=1}^{n} x_k^2 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1} x_k x_j\,, \tag{3.1.32}$$

by using (3.1.32) in (3.1.31) we obtain;

$$\begin{aligned}\int_0^t dW_s^2 &= \lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n} \Delta W_{t_k}^2\right)^2 + t^2 - 2t\sum_{k=1}^{n} \Delta W_{t_k}^2\right] \\ &= \lim_{n\to\infty} E\left[\sum_{k=1}^{n} \Delta W_{t_k}^4 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1} \Delta W_{t_k}^2 \Delta W_{t_j}^2 + t^2 - 2t\sum_{k=1}^{n} \Delta W_{t_k}^2\right]\end{aligned}, \tag{3.1.33}$$

now we know from the moments of Brownian's motion that;

$$\begin{aligned}&E[B_t^{2k+1}] = 0 \\ &E[B_t^{2k}] = (2k-1)!!\, t^k; k = 0,1,2,\cdots\end{aligned}, \tag{3.1.34}$$

where

$$n!! = n(n-2)(n-4). \tag{3.1.35}$$

So if that ;

$$\begin{aligned}E[B_t] &= 0 \\ E[B_t^2] &= t \\ E[B_t^3] &= 0 \\ E[B_t^4] &= 3t^2\end{aligned}, \tag{3.1.36}$$

using previous relationships, we find that;

$$E[\Delta W_t^4] = 3\Delta t^2, \quad E[\Delta W_t^2] = \Delta t, \tag{3.1.37}$$

Now from (3.1.33) and because the Brownian increments and disjoint interval are independent, we replace expected value of ΔW squared by Δt we obtain;

$$\int_0^t dW_s^2 = \lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n} \Delta W_{t_k}^2\right)^2 + t^2 - 2t\sum_{k=1}^{n} \Delta W_{t_k}^2\right]$$
$$= \lim_{n\to\infty} E\left[\sum_{k=1}^{n} \Delta W_{t_k}^4 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1} \Delta W_{t_k}^2 \Delta W_{t_j}^2 + t^2 - 2t\sum_{k=1}^{n} \Delta W_{t_k}^2\right], \qquad (3.1.38)$$
$$= \lim_{n\to\infty} \left[3\sum_{k=1}^{n} \Delta t_k^2 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1} \Delta t_k \Delta t_j + t^2 - 2t\sum_{k=1}^{n} \Delta t_k\right]$$

but $\sum_{k=1}^{n} \Delta t_k = t$ and $\Delta t = \frac{t}{n}$, so if that;

$$\int_0^t dW_s^2 = \lim_{n\to\infty} \left[3\sum_{k=1}^{n} \Delta t_k^2 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1} \Delta t_k \Delta t_j + t^2 - 2t^2\right]$$
$$= \lim_{n\to\infty} \left[3\sum_{k=1}^{n} \Delta t_k^2 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1} \Delta t_k \Delta t_j - t^2\right]$$
$$= \lim_{n\to\infty} \left[3\sum_{k=1}^{n} \left(\frac{t}{n}\right)^2 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1} \frac{t}{n}\frac{t}{n} - t^2\right] \quad , \qquad (3.1.39)$$
$$= \lim_{n\to\infty} \left[3n\frac{t^2}{n^2} + 2\frac{n(n-1)}{2}\frac{t^2}{n^2} - t^2\right]$$
$$= \lim_{n\to\infty} \left[3\frac{t^2}{n} + \left(1 - \frac{1}{n}\right)t^2 - t^2\right]$$
$$= [0 + (1-0)t^2 - t^2]$$
$$= 0$$

and we thus conclude that the integral of dW squared over an interval goes to the length of the interval;

$$\int_0^t d\,W_s^2 = t \Rightarrow dW_t^2 = dt, \qquad (3.1.40)$$

and this is what is meant by dW squared equal to dt.

Definition 3.10. Stochastic Integral. For any function $f = f(t)$ the stochastic integral is a function like $W = W(t)$, $t \in [0, T]$ which is given by,

$$W(t) = \int_0^t f(s)dB(s) = \lim_{n\to\infty} \sum_{k=1}^{N} f(t_{k-1})\big(B(t_k) - B(t_{k-1})\big), \qquad (3.1.41)$$

where $t_k = \frac{kt}{N}$.

The main types of stochastic integrals that appear in SDEs are:

1. Ito stochastic integral.
2. Stratonovich integral.

Definition 3.11. Ito's Integral. Ito was the inventor of the theory that describes movement due to random events, which is called the theory of stochastic differential equations, in 1942, as Ito began his work from scratch in reconstructing the stochastic integrals in addition to the theory of analysis associated with it. Ito continued and developed his thoughts on stochastic analysis after receiving his doctorate in 1945. Besides, Ito published many effective papers on this topic. Among these papers are "On a stochastic integral equation" (1946), "On the stochastic integral" (1948), "Stochastic differential equations in a differentiable manifold" (1950), "Brownian motions in a Lie group" (1950), and "On stochastic differential equations" (1951).

The *Ito stochastic integral* for the step process $G \in L^2(0,T)$ on the interval $(0,T)$ is given by the following form:

$$\int_0^T GdW \coloneqq \sum_{k=0}^{m-1} G_k\big(W(t_{k+1}) - W(t_k)\big), \qquad (3.1.42)$$

Definition 3.12. Stratonovich integral. The stochastic calculus that can be used as an alternative for Ito calculus was invented by Stratonovich. Besides, when looking at physical laws, Stratonovich calculus is considered completely natural. Additionally, Stratonovich integral appears in his stochastic calculus. Moreover, he solved the optimization problem other than non-linear filtering on the basis of his theory of Markov conditional processes, which published in his papers in 1959 and 1960. The Kalman-Bucy (linear) filter (1961) is a special case of Stratonovich's filter. In addition, he developed the value of information theory (1965). The other most popular stochastic integral besides Ito stochastic integral is, Stratonovich stochastic integral defined as

$$\int_0^T B(X,t) \circ dB(t) = \lim_{|P^n|\to 0} \sum_{k=0}^{m_n-1} B\left(\frac{X(t_{k+1}^n) - X(t_k^n)}{2}, t_k^n\right)\left(W(t_{k+1}^n) - W(t_k^n)\right)$$

provided this limit exists in $L^2(\Omega)$ for all sequences of partitions P^n, with

$|P^n| \to 0$. Here $X(.)$ a stochastic process with values in R^n. The symbol "∘" is used to denote that the integral is the Stratonovich integral. Two integrals find use in different areas. The Ito integral is used mostly in the fields of mathematics and finance because of martingale property, the Stratonovich integral is more popular in physics because of the limit of smooth noise argument. In particular, two integrals are mathematically equivalent in the following sense; any Stratonovich process

$$dX_t = f(X_t,t)dt \pm \sigma(X_t,t) \circ dB_t$$

has an equivalent Ito process with identical solutions, which is given by

$$dX_t = f(X_t,t)dt \neq \sigma(X_t,t)dB_t \pm \frac{1}{2}\frac{\partial\sigma}{\partial x}(X_t,t)a(X_t,t)dt.$$

This formula holds in both directions. So, if we already have a well-defined SDE, either in the Ito or Stratonovich sense, then we can convert between the two conventions arbitrarily, depending on which properties we feel are more convenient for the problem at hand. Notice that the conversion formula basically comes down to a modification of the drift (dt) term.

3.1.3. Stochastic differential equations

If one or more of the terms of a differential equation is a stochastic process, then this equation is called a stochastic differential equation (SDE). In addition, the resulting solution is also a stochastic process in itself. There are various phenomena such as thermal fluctuations or the fluctuation of stock prices that subject to randomness. For their modeling, SDE is used. We also know that from the Brownian motion (or Wiener process) white noise is derived which would normally be incorporated into SDEs.

Systems in many branches of science and industrial sectors are often affected by different types of environmental noise. Therefore, in order for the mathematical modeling of many problems to be closer to reality, either the coefficients of the terms of the differential equations should contain randomness or a forcing term containing randomness must be added to the equation.

For an example, let's take the population growth model given in (Mao, 2007) and (Øksendal, 2003) as follows

$$\frac{dN(t)}{dt} = f(t)N(t), N(0) = N_0, \tag{3.1.43}$$

where $N(t)$ is the population size at time $t, f(t)$ is related with deterministic growth rate at time t, and $\frac{dN(t)}{dt}$ is the rate of change in population size. Consider that the function $f(t)$ is not known completely but depends on some random external effects. In other words, it is written in the form;

$$f(t) = r(t) + \sigma(t)\text{"noise"}. \tag{3.1.44}$$

Here, the behavior of the term "noise" is not completely known, only the probability distribution is known. Let us consider that the function $r(t)$ is non-random and replaced with $f(t)$ in the previous population growth model, we find that

$$\frac{dN(t)}{dt} = r(t)N(t) + \sigma(t)N(t)\text{"noise"}. \tag{3.1.45}$$

It can be written in the integral form as follows

$$N(t) = N_0 + \int_0^t r(s)N(s)d\,s + \int_0^t \sigma(s)N(s)\text{"noise"}d\,s. \tag{3.1.46}$$

Whereas the mathematical interpretation of the term "noise" in the literature is derived from Brownian motion (Wiener process) and is denoted by $\frac{dW(t)}{dt}$. Based on the above, the term "noise"dt can be expressed as "noise"$dt = \dot{W}(t)dt = dW(t)$ and

$\int_0^t \sigma(s)N(s)$"noise"$\,ds = \int_0^t \sigma(s)N(s)\,dW(s)$. The following form of (3.1.46) Called the integral formula

$$N_t = N_0 + \int_0^t r(s)N(s)d\,s + \int_0^t \sigma(s)N(s)dW(s)\,, \qquad (3.1.47)$$

and the differential form of (3.1.46) is

$$dN_t = r(t)N(t)dt + \sigma(t)N(t)dW(t). \qquad (3.1.48)$$

Now if we include the parameters ϕ, the continuous time t and the variable N_t, then the differential equation (3.1.48) in a general form can be written as follows

$$dN(t) = b(N(t), t, \phi)dt + \sigma(N(t), t, \phi)dW(t), \qquad (3.1.49)$$

or in integral form as

$$N(t) = N_0 + \int_{t_0}^t b(s, N_s)d\,s + \int_{t_0}^t \sigma(s, N_s)dW_s\,, \qquad (3.1.50)$$

whereas the drift and diffusion coefficients are b and σ, respectively, and the above differential equation is called the Ito stochastic differential equation. N_0 is called the initial value of the random variable at moment t_0 and the stochastic process N_t is called the solution for (3.1.49) or (3.1.50).

3.1.4. Existence and uniqueness theorem for SDE

Let $A, B: R \times [0, T] \to R$ be continuous functions satisfying (Øksendal, 2003),

$$\text{Lipschitz condition } \begin{cases} |A(x,t) - A(y,t)| \le \delta|x-y| \\ |B(x,t) - B(y,t)| \le \delta|x-y| \end{cases} \forall x, y \in R, t \in [0, T]\,, \qquad (3.1.51)$$

$$\text{Linear growth condition } \begin{cases} |A(x,t)| \le \delta(1+|x|) \\ |B(x,t)| \le \delta(1+|x|) \end{cases} \forall x \in R, t \in [0, T]\,, \qquad (3.1.52)$$

for some constant δ. Let X be a random variable independent of the filtration $\mathcal{F}_0$ relative to a standard Brownian motion $\{W_t\}$, such that

$$E|X|^2 < \infty,$$

Then there exists a unique solution $X_t \in L^2(0,T)$ of the SDE

$$\begin{cases} dX_t = A(X_t,t)dt + B(X_t,t)dW_t \\ X_0 = X \end{cases}$$

3.1.5. Wick-product

In 1991 Lindstrom et al. (Lindstrøm et al., 1991) proved that there is a close relationship between the deterministic differential equation of the form

$$\frac{dx_t}{dt} = b(x_t) + s(x_t) \cdot \sum_k x_k(t) z_k\,, \tag{3.1.53}$$

and the Ito-Skorohod stochastic differential equation of the form

$$dX_t = b^{\diamond}(X_t)dt + \sigma^{\diamond}(X_t)\delta. \tag{3.1.54}$$

Here $b^{\diamond}$, $\sigma^{\diamond}$ show Wick versions of functions b and σ, respectively. The connection is achieved by converting the stochastic process X_t taken from L^2 into the analytical function $\mathcal{H}(X_t)(z_1, z_2, \cdots)$ with the help of the $\mathcal{H}$ Hermite transformation and its $\mathcal{H}^{-1}$ inverse. The Wick product was firstly defined by G.C. Wick (Wick, 1950) in the scope of his work on quantum theory in 1950, and a similar concept was later introduced in the field of probability theory by Hida and Ikeda in 1965. Then the Wick product has become an important tool in the study of stochastic differential equations. To establish a connection between stochastic differential equations and deterministic ones, we need to use Hermite transform which converts the Wick products into ordinary products. The basic idea of using the Hermite transform is to make a correspondence between the elements of the space of stochastic distributions and a space of analytic functions of complex variables. In the white noise space, the name of the Wick product is given over the standard multiplication of two items. On the other hand, in a white noise space the Wick product is unstable so we must look at larger (or smaller) space where it is stable. The so-called weighted stochastic spaces whose elements are characterized by their Wiener chaos coefficients which include Hida and Kondratiev spaces of stochastic test functions and stochastic distributions are the required spaces. Below we will give a brief description of them.

3.1.6. Preliminaries on white noise analysis

The Schwartz space $\mathcal{S}(R^d)$ consists of rapidly decreasing functions $f: \mathbb{R}^d \to \mathbb{C}$ such that $f \in C^{\infty}(R^d)$ and for all multi-indices $\alpha \geq 0$ and $\beta \geq 0$ the function mapping x to $x^{\alpha}\partial^{\beta} f(x)$ is bounded on R^d (Adams and Fournier, 2003). The space of tempered distributions $\mathcal{S}'(R^d)$ equipped with a weak-star topology is dual of $\mathcal{S}(R^d)$. Tempered distributions can be thought of as distributions that do not grow faster than a polynomial at infinity. By the Bohner-Minlos theorem there exists a unique probability measure μ on Borel subsets $\mathcal{B}\left(\mathcal{S}'(R^d)\right)$ of $\mathcal{S}'(R^d)$ which forms a white noise probability space $\left(\mathcal{S}'(R^d), \mathcal{B}\left(\mathcal{S}'(R^d)\right), \mu\right)$. Throughout this work, we also use the spaces $(\mathcal{S})$ and $(\mathcal{S})^*$ which provide a suitable environment for stochastic differential equations and are named as Hida test function space and Hida distribution space, respectively. Hida distribution and test function spaces are subspaces of $L^2(R)$ in some respects corresponding to Schwartz subspaces of $L^2(R)$. Let the Hermite polynomial $h_n(x)$ of order n be defined by

$$h_n(x) = (-1)^{n \mathrm{e}^{\frac{x^2}{2}}} \frac{d^n}{dx^n}\left(\mathrm{e}^{-\frac{x^2}{2}}\right), \quad n = 0,1,2,\dots, \tag{3.1.55}$$

and the nth Hermite function $\zeta_n(x)$ be defined by

$$\zeta_n(x) = \pi^{-1/4}\left((n-1)!\right)^{-1/2 \mathrm{e}^{-\frac{1}{2}x^2}} h_{n-1}\left(\sqrt{2}x\right), \quad n \geq 1. \tag{3.1.56}$$

Then $\zeta_n(x) \in \mathcal{S}(R^d)$ is an eigenfunction for the operator $A = -\left(\frac{d}{dx}\right)^2 + x^2 + 1$, and $\{\zeta_n\}_{n\geq 1}$ constitutes an orthonormal basis for $L^2(R)$. Therefore, the family of tensor products $\zeta_{\alpha} = \zeta_{(\alpha_1,\dots,\alpha_d)} = \zeta_{\alpha_1} \otimes \cdots \otimes \zeta_{\alpha_d} (\alpha \in N^d)$ forms an orthogonal basis for $L^2(R)$, where $\alpha = (\alpha_1, \dots, \alpha_d)$ denote $d-$dimensional multi-indices with $\alpha_1, \dots, \alpha_d \in N$. Assume that the family of i -th multi-index $\alpha^{(i)} = \left(\alpha_1^{(i)}, \dots, \alpha_d^{(i)}\right)$ is given in some fixed ordering

$$i < j \Rightarrow \alpha_1^{(i)} + \cdots + \alpha_d^{(i)} \leq \alpha_1^{(j)} + \cdots + \alpha_d^{(j)}$$

i.e., the $\left\{\alpha^{(j)}\right\}_{j=1}^{\infty}$ is formed in an increasing order. Let us define

$$\eta_i = \zeta_{\alpha^{(i)}} = \zeta_{\alpha_1^{(i)}} \otimes \cdots \otimes \zeta_{\alpha_d^{(i)}}, \quad i \geq 1.$$

To take the multi-indices of arbitrary length, we consider multi-indices of elements of the space $(N_{\mathbb{0}}^{\mathrm{N}})_c$, of all sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ with elements $\alpha_i \in N_{\mathbb{0}}$ and with compact support. Let $J = (N_{\mathbb{0}}^{N})_c$, and $\alpha \in J$. Before defining Kondratiev and Hida spaces, we will give the following useful theorem named Wiener-Ito chaos expansion which provides an orthonormal expansion using Hermite polynomials.

Theorem 3.2. ***.*** *(Wiener-Ito chaos expansion). Holden et al. (2010). Every element f in L^2 has a unique expansion*

$$f(\omega) = \sum_{\alpha} c_\alpha H_\alpha(\omega),$$

where $c_\alpha \in R^n$. By H_α, we define the random variable

$$H_\alpha(\omega) = \prod_{i=1}^{\infty} h_{\alpha_i}(\langle \omega, \eta_i \rangle), \quad \omega = (\omega_1, \dots, \omega_m) \in \mathcal{S}'(R^d).$$

Moreover, let us note that $\{H_\alpha\}_{\alpha \in J}$ forms an orthogonal basis in $L^2\left(\mathcal{S}'(R^d)\right)$, and has norm expression

$$|H_\alpha|^2_{L^2\left(\mathcal{S}'(R^d)\right)} = \alpha! := \alpha_1!\, \alpha_2! \dots$$

Now, we will define the Kondratiev spaces of stochastic test and distribution spaces.

Definition 3.13. Space of the Kondratiev test functions. For $k = 1,2, \cdots and -1 \leq \rho \leq 1$, let

$$(\mathcal{S})_{\rho,k} = \left\{ f \in L^2(\mu) : f(\omega) = \sum_{\alpha} c_\alpha H_\alpha(\omega), c_\alpha \in R \right\},$$

such that

$$|f|^2_{(\mathcal{S})_{\rho,k}} := \sum_{\alpha} (\alpha!)^{1+\rho} c_\alpha^2 (2N)^{k\alpha} < \infty,$$

where

$$(2N)^{k\alpha} = \prod_{i,j=1}^{m} (2(i-1)m + j)^{\alpha_{ij}}, \text{ if } \alpha = \left(\alpha_{ij}\right)_{1 \le i,j \le m}.$$

The space of Kondratiev test functions $(\mathcal{S})_{\rho}$, is defined by

$$(\mathcal{S})_{\rho} = \bigcap_{k=1}^{\infty} (\mathcal{S})_{\rho,k}$$

Definition 3.14. Space of Hida distributions. The space of the Hida distributions can be described in simple terms as follows:

- The space of Hida distributions, $(\mathcal{S})_{-\rho}$, is defined by

$$(\mathcal{S})_{-\rho} = \bigcup_{k=1}^{\infty} (\mathcal{S})_{-\rho,k}$$

- We have

$$(\mathcal{S})_{\rho} \subset L^2(\mu) \subset (\mathcal{S})_{-\rho}$$

Definition 3.15. Wick product. Let $F = \sum_{\alpha} a_{\alpha} H_{\alpha}$ and $G = \sum_{\beta} a_{\beta} H_{\beta}$ be the two elements taken from $(\mathcal{S})_{-1}^{n}$, $a_{\alpha}, b_{\alpha} \in R^{n}$. In this case, the Wick product of F and G is denoted by $F \diamond G$ and identified by

$$F \diamond G = \sum_{a,\beta} a_{a} b_{\beta} H_{a+\beta} = \sum_{\gamma} c_{\gamma} H_{\gamma}.$$

Where $c_{\gamma} = \sum_{\alpha+\beta=\gamma} a_{\alpha} b_{\beta}$.

The following basic algebraic properties of the Wick product follow directly

from the definition.

- (Commutative law) $F, G \in (\mathcal{S})_{-1}^{n} \Rightarrow F \diamond G = G \diamond F$.
- (Associative law) $F, G, K \in (\mathcal{S})_{-1}^{n} \Rightarrow F \diamond (G \diamond K) = (F \diamond G) \diamond K$.
- (Distributive law)$F, G, K \in (\mathcal{S})_{-1}^{n} \Rightarrow F \diamond (G + K) = F \diamond G + F \diamond K$.

Definition 3.16. Hermite transform. Let $F = \sum_{\alpha} b_{\alpha} H_{\alpha}$ taken from $(\mathcal{S})_{-1}^{n}$, $a_{\alpha} \in R^{n}$. In this case, the Hermite transform of F, which denoted by $\mathcal{H}F$ or $\tilde{F}$, is defined by

$$\mathcal{H}F(z) = \tilde{F}(z) = \sum_{\alpha} b_{\alpha} z^{\alpha} \in C^{n},$$

where $z = (z_1, z_2, \ldots) \in C^{N}$ and $z^{\alpha} = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_j^{\alpha_j}$ if $\alpha = (\alpha, \alpha, \ldots) \in \mathcal{J}$ where $z_i^0 = 1$

3.1.7. Galilean transform

The set of equations that relate the coordinates of space and time in classical physics for two systems moving at a constant speed relative to each other are called Galilean transformations. These transformations were mentioned in the paper of Wadati and Akutsu (Wadati and Akutsu, 1984) to convert the stochastic PDE into a deterministic PDE. In this thesis, we will use these transformations for solving stochastic differential equations. For an example and clarification about this transformation, we will review an example from Wadati’s article (Wadati and Akutsu, 1984). *Example.* By using the following Galilean transform

$$\begin{aligned} &u(x,t) = U(X,t) + W(t) \\ &X = x + m(t) \\ &m(t) = 6\int_0^t W(t')dt' \quad , \\ &W(t) = \int_0^t \eta(s)ds \end{aligned} \tag{3.1.58}$$

we can transform the following stochastic equation

$$u_t - 6uu_x + u_{xxx} = \eta(t), \tag{3.1.59}$$

to the following deterministic form

$$U_t(x,t) - 6U(x,t)U_x(x,t) + U_{xxx}(x,t) = 0. \tag{3.1.60}$$

3.2. Basic Concepts on The Nonlinear Methods

Throughout the thesis we use tanh, extended tanh and F-expansion methods. The aim of this section is to provide basic information on these methods.

The methods of tanh and extended tanh methods mainly lead to transforming of a traveling wave as $u(x,t) = u(x - \nu t) = U(\xi)$, where $U(\xi)$ is the wave solution traveling at speed ν. Then these methods assume a priori that travelling wave solutions may be expressed in terms of the function tanh. In the following, we explain these methods.

3.2.1. Preliminaries of the method of tanh

Firstly, we start by considering the equation of nonlinear partial differential in a general form with two independent variables t and x describing the waveform $u(x,t)$

$$P(u, u_t, u_x, u_{xx}, \cdots) = 0, \tag{3.2.1}$$

where P in its arguments represents a polynomial and we want to know that if the travelling waves are solutions of (3.2.1), or not.

1. As a first step, we introduce the following transformation to produce the traveling wave solution for Eq. (3.2.1)

$$u(x,t) = V(z), z = k(x - ct), \tag{3.2.2}$$

where both constants k and c are determined later. As a result of this transformation, the derivatives are changed into

$$\begin{gathered} \frac{\partial}{\partial t} = -kc\frac{d}{dz}, \frac{\partial}{\partial x} = -k\frac{d}{dz} \\ \frac{\partial^2}{\partial x^2} = k^2\frac{d^2}{dz^2}, \frac{\partial^3}{\partial x^3} = k^3\frac{d^3}{dz^3}, \dots, \end{gathered} \tag{3.2.3}$$

and so on for the other derivates. Substitution of (3.2.3) into (3.2.1) yields the following form of the ODE

$$P(V, V', V'', V''', \dots) = 0. \tag{3.2.4}$$

2. In this step there are two options, for more details see (Malfliet, 2004).

   a) *With zero boundary condition.* In some cases, boundary conditions may be imposed a priori, for example, if we analyze the problems with vanishing tail or front, we implement conditions such as

$$V(z) \to 0, \quad \frac{d^nV(z)}{dz^n} \to 0, \quad (n = 1,2, \dots) \quad for \quad z \to \pm\infty. \tag{3.2.5}$$

In this case, the constants resulting from the process of integrating the differential equation with respect to the derivative z are selected as zeros (Malfliet, 1992, 2004). Then we obtain a simplified ODE.

b) *Without boundary conditions.* We move to the **step (3)** without integrating the ODE.

3. Now we change the independent variable z to another new independent variable as follows

$$Y = tanh(z) \quad or \quad Y = coth(z). \tag{3.2.6}$$

As a result of the previous transformation, we get a change in the derivatives as follows

$$\begin{aligned} \frac{d}{dz} &= (1 - Y^2)\frac{d}{dY} \\ \frac{d^2}{dz^2} &= (1 - Y^2)\left(-2Y\frac{d}{dY} + (1 - Y^2)\frac{d^2}{dY^2}\right) \\ \frac{d^3}{dz^3} &= (1 - Y^2)\left((6Y^2 - 2)\frac{d}{dY} - 6Y(1 - Y^2)\frac{d^2}{dY^2} + (1 - Y^2)^2\frac{d^3}{dY^3}\right). \end{aligned} \tag{3.2.7}$$

4. Applying the previous steps, the solution will be written as a series of powers of Y as follows

$$V(z) = \sum_{s=0}^{M} a_s Y^s , \tag{3.2.8}$$

where M is a parameter determined in the next step by using homogeneous balance. Then by using both of (3.2.7) and (3.2.6) in ODE (3.2.4) we obtain an equation in the powers of Y .

5. The balancing integer term M can be calculated using the principle of homogeneous balance after using (3.2.4) in (3.2.6), as follows:

$$D\left[\frac{d^qV}{dz^q}\right] = M + q$$

$$D\left[V^r\left(\frac{d^qV}{dz^q}\right)^s\right] = Mr + s(q + M)$$

Therefore, the value of M in Eq. (3.2.8) is found.

6. Reaching this step means that the integer positive number M is known. If we make the coefficients of the powers of Y in the resulting equation from step 4 equal to zero we obtain a system of algebraic equations with the unknown parameters a_s, $(s = 0, 1, \dots, M)$. By solving this system, we obtain the required parameters. Finally, using (3.2.8), we obtain the analytical solution in closed form of (3.2.1).

3.2.2. Preliminaries of the method of extended-tanh

In this method, the same algorithm is followed in the tanh method except for step 4, the expansion is replaced by the next expansion of the solution $u(x, t)$

$$V(z) = \sum_{s=0}^{M} a_s Y^s + \sum_{s=1}^{M} a_{-s} Y^{-s} \,. \tag{3.2.9}$$

Likewise, in the tanh method, we obtain the parameter M by applying the principle of homogeneous balance between the higher order nonlinear and linear terms. Then we replace (3.2.9) in ODE (3.2.4) and proceed as suggested with the method of tanh.

3.2.3. Preliminaries of the F-expansion method

A brief description of the essential steps of the F-expansion method will be provided here since the method was explained in several papers (Wang and Li, 2005; Zhou et al., 2003). Before applying the F-expansion method, the Wick-type equation will be transformed to an ordinary products equation by using the Hermite transform. So, the first step will be the Hermite transformation.

1. The Wick-type equation.

$$A^{\diamond}(t, x, \partial_t, \nabla_x, U, \omega) = 0, \tag{3.2.10}$$

will be transformed to the following partial differential equation (PDE)

$$\tilde{A}\big(t, x, \partial_t, \nabla_x, \widetilde{U}, z_1, z_2, \dots\big) = 0, \tag{3.2.11}$$

by use of the Hermite transform. This step replaces the real-valued function with a complex-valued function. Now, we have to solve a deterministic PDE with complex coefficients.

2. The deterministic PDE with complex coefficients (3.2.11) will be converted into an ordinary differential equation (ODE) by considering the transformation of the form $\widetilde{U}(t,x,z) = u(t,x,z) = u(\zeta), \zeta = f(t,x)x + g(t,x)$. The converted equation reads as follows

$$A\left(u, u_\zeta, u_{\zeta\zeta}, \dots\right) = 0. \tag{3.2.12}$$

3. We assume that the solution of Eq. (3.2.12) can be expressed in the form of finite series as follows

$$u(\zeta) = \sum_{i=0}^{n} a_i(t,z)F^i(\zeta)\,, \tag{3.2.13}$$

where the function $F(\zeta)$ satisfies the following elliptic equation of first kind

$$F_\zeta^2 = A_1 + A_2F^2 + A_3F^4, \tag{3.2.14}$$

Eq. (3.2.14) has 24 Jacobian elliptic function solutions given in Table 3.3.

Table 3.3. The 24 Jacobian elliptic function solutions of Eq. (3.2.14).

Case	A_1	A_2	A_3	$F(\zeta)$
1	1	$-(1+m^2)$	m^2	$sn(\zeta)$
2	1	$-(1+m^2)$	m^2	$cd(\zeta)$
3	$1-m^2$	$2m^2-1$	$-m^2$	$cn(\zeta)$
4	m^2-1	$2-m^2$	-1	$dn(\zeta)$
5	m^2	$-\ (1+m^2)$	1	$ns(\zeta)$
6	m^2	$-(1+m^2)$	1	$dc(\zeta)$
7	$-m^2$	$2m^2-1$	$1-m^2$	$nc(\zeta)$
8	-1	$2-m^2$	$1-m^2nd(\zeta)$	$nd(\zeta)$
9	1	$2-m^2$	$1-m^2$	$sc(\zeta)$
10	1	$2m^2-1$	$-m^2(1-m^2)$	$sd(\zeta)$
11	$1-m^2$	$2-m^2$	1	$cs(\zeta)$
12	$-m^2(1-m^2)$	$2m^2-1$	1	$ds(\zeta)$
13	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$ns(\zeta) \pm ds(\zeta)$
14	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$sn(\zeta) \pm \mathrm{i}cn(\zeta)$
15	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\sqrt{1-m^2}sd(\zeta) \pm cd(\zeta)$

Table 3.3. (Continue) 1.

Case	A_1	A_2	A_3	$F(\zeta)$
17	$\frac{1}{4}$	$\frac{1-m^2}{2}$	$\frac{1}{4}$	$msn(\zeta) \pm \mathrm{i}dn(\zeta)$
18	$\frac{1}{4}$	$\frac{1-m^2}{2}$	$\frac{1}{4}$	$ns(\zeta) \pm cs(\zeta)$
19	$\frac{1}{4}$	$\frac{1-m^2}{2}$	$\frac{1}{4}$	$\sqrt{1-m^2}sc(\zeta) \pm dc(\zeta)$
20	$\frac{m^2-1}{4}$	$\frac{m^2+1}{2}$	$\frac{m^2-1}{4}$	$msd(\zeta) \pm nd(\zeta)$
21	$\frac{1-m^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1-m^2}{4}$	$nc(\zeta) \pm sc(\zeta)$
22	$\frac{1+4m^2-m^4}{4}$	$\frac{m^2+1}{2}$	$-\frac{1}{4}$	$mcn(\zeta) \pm dn(\zeta)$
23	$\frac{(1-m^2)^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1}{4}$	$ds(\zeta) \pm cs(\zeta)$
24	$\frac{m^4(1-m^2)}{2(2-m^2)}$	$\frac{2(1-m^2)}{m^2-2}$	$\frac{1-m^2}{2(2-m^2)}$	$dc(\zeta) \pm \sqrt{1-m^2}nc(\zeta)$

$*$ m is the module of Jacobian elliptic function.

4. To specify the parameter n of step 3, the highest order nonlinear term and highest order derivative term in Eq. (3.2.12) will be balanced. Substitution of n into Eq. (3.2.13) yields the solution of Eq. (3.2.12) as follows

$$u(\zeta) = a_0(t,z) + a_1(t,z)F(\zeta) + \cdots + a_n(t,z)F(\zeta)^n. \tag{3.2.15}$$

5. The next step is substitution of Eqs. (3.2.14) and (3.2.15) into Eq. (3.2.12). Then setting the coefficients of all powers of F^i, xF^i and $F_\zeta F^i$ of the resulting equation to zero gives a set of algebraic equations. By solving the set of algebraic equations, the parameters $f, g, a_i (i = 0,1, \dots, n)$ can be obtained explicitly.
6. By putting the parameters obtained in the previous step into Eq. (3.2.15) and $\zeta = f(t,x)x + g(t,x)$ we obtain the general formal solution $u(\zeta) = u(t,x,z)$ of Eq. (3.2.11).
7. Obtain general formal solution for Eq. (3.2.10). Taking the inverse Hermite transformation of $u(t,x,z)$ obtained in Step 6, i.e., $U(t,x) = \mathcal{H}\big(u(t,x,z)\big)$.We

deduce $U(t,x)$ which is a general formal solution of Wick-type stochastic Eq. (3.2.10).

8. Derive the solutions of Jacobian elliptic function for Eq. (3.2.10). Replacing $A_1, A_2, A_3, F(\zeta)$ in $U(t,x)$ obtained in step 7 with corresponding values in the Table 3.3, we obtain a series of solutions of Jacobian elliptic function for Wick-type stochastic Eq. (3.2.10).

4. RESULTS

In this chapter, the method of tanh and extended tanh, the first two of the proposed methods for solving the evolution equations, are applied to a set of famous equations. As a first step, the search is for the appropriate Galilean transformation that transfers the stochastic equation into the deterministic case. Then methods of tanh and extended-tanh are used here to find solutions to the resulting equation, and then by replacing the Galilean transformation in the resulting equations, closed solutions of the studied equations are obtained. Finally, we visualize some solutions using computer programs to show the effect of stochastic terms on the solution.

4.1. Using Galilean Transform for Solving the Stochastic KdV–Burgers Equation Via The Method of Tanh

The Korteweg-de Vries-Burgers equation is often used in the description of wave processes in dissipative-dispersive systems in many areas of physics (Kudryashov, 1991). Let us start with the following equation

$$U_t + UU_X - BU_{XX} + RU_{XXX} = \eta(T), \tag{4.1.1}$$

we call Eq. (4.1.1) a stochastic Korteweg de Vries-Burger's (KdVB) equation. Here inhomogeneous term $\eta(T)$ stands for external noise and both of X and T point to partial differentiations with respect to X and T, respectively. One simply applies the following Galilean transformation

$$U(X,T) = u(x,t) + W(T), x = X + m(t), t = T, \tag{4.1.2}$$

$$m(T) = -\int_0^T W(T')dT', \quad W(T) = \int_0^T \eta(T')dT', \tag{4.1.3}$$

to convert the stochastic Kortewegde Vries-Burger's equation into its deterministic counterpart

$$u_t + uu_x - Bu_{xx} + Ru_{xxx} = 0. \tag{4.1.4}$$

In the following sections, both the method of tanh and extended tanh will be applied, respectively, to the stochastic Korteweg de Vries-Burger's equation to develop solitary wave solutions. It should be noted that the method of extended tanh gives further solitary wave solutions. Boundary conditions can be applied a priori to reduce unnecessary calculations.

4.1.1. The method of tanh with zero boundary condition

Introducing the following wave variable $u(x,t) = V(z), z = k(x-ct)$ carries the Korteweg de Vries-Burger's equation (4.1.4) into an ODE "*with zero boundary condition*", i.e, for $V(z) \to 0$, $\frac{dnV(z)}{dz} \to 0$, $\frac{d^2V(z)}{dz^2} \to 0$ as $z \to \pm\infty$,

$$-ckV + \frac{1}{2}V^2 - Bk^2V' + Rk^3V'' = 0. \quad (4.1.5)$$

Balancing $V''$with V^2 in (4.1.5) lead to

$$M + 2 = 2M. \quad (4.1.6)$$

Therefore

$$M = 2. \quad (4.1.7)$$

Hence the next step of the method of tanh gives the following finite expansion

$$V(z) = a_0 + a_1Y + a_2Y^2, Y = tanh(z). \quad (4.1.8)$$

From the two relations (4.1.8) and (4.1.5), we get the following system of algebraic equations for a_0, a_1, a_2, k and c after collecting the coefficients $Y^S, (S = 0,1,2, \dots, 4)$ with each other and equating them to zero

$$\begin{aligned}
&Y^0 .: \quad \tfrac{1}{2}ka_0^2 + 2a_2Rk^3 - Bk^2a_1 - cka_0 = 0 \\
&Y^1 .: \quad -2Rk^3a_1 - 2Bk^2a_2 - cka_1 + ka_0a_1 = 0 \\
&Y^2 .: \quad ka_0a_2 + \frac{1}{2}ka_1^2 - 8a_2Rk^3 + Bk^2a_1 - cka_2 = 0 \qquad (4.1.9) \\
&Y^3 .: \quad 2Rk^3a_1 + 2Bk^2a_2 + ka_1a_2 = 0 \\
&Y^4 .: \quad \tfrac{1}{2}ka_2 + 6a_2Rk^3 = 0
\end{aligned}$$

Now, using one of the symbolic calculation programs to solve the above algebraic system, we get the following two cases of solutions:

1.

$$c = -\frac{6B^2}{25R}, \quad k = \frac{B}{10R}, \quad a_0 = -\frac{3B^2}{25R}, \quad a_1 = -\frac{6B^2}{25R}, \quad a_2 = -\frac{3B^2}{25R}. \quad (4.1.10)$$

2.

$$c = \frac{6B^2}{25R}, \quad k = \frac{B}{10R}, \quad a_0 = \frac{9B^2}{25R}, \quad a_1 = -\frac{6B^2}{25R}, \quad a_2 = -\frac{3B^2}{25R}. \tag{4.1.11}$$

Solutions of the case 1. Substituting the following soliton solutions (4.1.10) into (3.2.8) and using (3.2.6), (3.2.2), we obtain the solution;

$$u_1(x,t) = -\frac{3B^2}{25R}(1 + \tanh[\phi])^2, \tag{4.1.12}$$

and

$$u_2(x,t) = -\frac{3B^2}{25R}(1 + \coth[\phi])^2, \tag{4.1.13}$$

where $\phi = \frac{B}{10R}\left(\frac{6B^2}{25R}t + x\right)$. We should note that in the above solution if $\phi \to -\infty$ then $tanh[\phi] \to -1$ and the solution u_1 satisfies boundary conditions. From (4.1.12)- (4.1.13) and Eqs. (4.1.2)-(4.1.3), we arrive at a set of exact stochastic solutions of Eq. (4.1.1), which are simplified as follows:

$$U_1(X,T) = -\frac{3B^2}{25R}(1 + \tanh[\phi_1(X,T)])^2 + W(T), \tag{4.1.14}$$

and

$$U_2(X,T) = -\frac{3B^2}{25R}(1 + \coth[\phi_2(X,T)])^2 + W(T), \tag{4.1.15}$$

where

$$\phi_1(X,T) = \phi_2(X,T) = \left[\frac{B}{10R}\left(\frac{6B^2}{25R}T + X - \int_0^T W(T')dT'\right)\right].$$

Solutions of the Case 2. Substituting the solutions (4.1.11) into (3.2.8) and using (3.2.6), (3.2.2), we arrive to the following soliton solutions;

$$u_3(x,t) = -\frac{3B^2}{25R}\left(1 + \tanh\left[\frac{B}{10R^2}\left(\frac{6B^2}{25R}t - x\right)\right]\right)\left(\tanh\left[\frac{B}{10R^2}\left(\frac{6B^2}{25R}t - x\right)\right] - 3\right), \tag{4.1.16}$$

and

$$u_4(x,t) = -\frac{3B^2}{25R}\left(1+\coth\left[\frac{B}{10R^2}\left(\frac{6B^2}{25R}t-x\right)\right]\right)\left(\coth\left[\frac{B}{10R^2}\left(\frac{6B^2}{25R}t-x\right)\right]-3\right). \quad (4.1.17)$$

We should note that in the above solution if we take $\phi = \frac{B}{10R^2}\left(x-\frac{6B^2}{25R}t\right)$ then $\tanh\phi \to 1$ when $\phi \to \infty$, and the solution $u_3(x,t) = \frac{3B^2}{25R}(1-\tanh\phi)(3-\tanh\phi)$ satisfies boundary conditions. From (4.1.16)- (4.1.17) and Eqs. (4.1.2)- (4.1.3), we arrive at a set of exact stochastic solutions of Eq. (4.1.1), we arrive at a set of exact stochastic solutions of Eq. (4.1.1), which are simplified like this:

$$U_3(X,T) = -\frac{3B^2}{25R}(1+\tanh[\phi_3(X,T)])(\tanh[\phi_3(X,T)]-3)+W(T), \qquad (4.1.18)$$

and

$$U_4(X,T) = -\frac{3B^2}{25R}(1+\coth[\phi_4(X,T)])(\coth[\phi_4(X,T)]-3)+W(T), \qquad (4.1.19)$$

where

$$\phi_3(X,T) = \phi_4(X,T) = \left[\frac{B}{10R^2}\left(\frac{6B^2}{25R}T-X+\int_0^T W(T')dT'\right)\right]$$

4.1.2. The method of tanh without boundary condition

Introducing the following wave variable $u(x,t) = V(z), z = k(x-ct)$ carries the Kortewegde Vries-Burger's equation (4.1.4) into an ODE with " *no boundary conditions*".

$$-ckV' + kVV' - Bk^2V'' + Rk^3V''' = 0. \qquad (4.1.20)$$

Balancing V''' with $V'V$ in (4.1.20) gives

$$M + 3 = M + 1 + M. \qquad (4.1.21)$$

So that

$$M = 2. \qquad (4.1.22)$$

Hence the next step of the method of tanh gives the following finite expansion

$$V(z) = a_0 + a_1 Y + a_2 Y^2, Y = tanh(z). \qquad (4.1.23)$$

From the two relations (4.1.23) and (4.1.20), we get to the following system of algebraic equations for a_0, a_1, a_2, k and c after collecting the $Y^S, (S = 0,1,2, \dots ,5)$ coefficients to each other and equating them with zero

$$Y^0 .: \quad -2Rk^3 a_1 - 2a_2 Bk^2 - cka_1 + ka_0 a_1 = 0$$

$$Y^1 .: \quad -16Rk^3 a_2 + 2Bk^2 a_1 - 2cka_2 + 2ka_0 a_2 + ka_1^2 = 0$$

$$Y^2 .: \quad 8Rk^3 a_1 + 8Ba_2 k^2 + cka_1 - ka_0 a_1 + 3ka_1 a_2 = 0$$

$$Y^3 .: \quad 40Rk^3 a_2 - 2Bk^2 a_1 + 2cka_2 - 2ka_0 a_2 - ka_1^2 + 2ka_2^2 = 0, \qquad (4.1.24)$$

$$Y^4 .: \quad -6Rk^3 a_1 - 6Ba_2 k^2 - 3ka_1 a_2 = 0$$

$$Y^5 .: \quad -24Rk^3 a_2 - 2ka_2^2 = 0.$$

Now, using one of the symbolic calculation programs to solve the above algebraic system, we get the following solution:

$$c = -\frac{3B^2}{25R} + a_0, \quad k = \frac{B}{10R}, \quad a_0 = a_0, \quad a_1 = -\frac{6B^2}{25R}, \quad a_2 = -\frac{3B^2}{25R}. \qquad (4.1.25)$$

Substituting the solutions (4.1.25) into (4.1.23) and using (3.2.6),(3.2.2), we arrive to the following solitons solution which may be interpreted as a dark soliton solution:

$$u(x,t) = a_0 - \frac{3B^2}{25R}\left(\tanh^2\left[\frac{B}{10R}\left(\left(\frac{3B^2}{25R} - a_0\right)t + x\right)\right] + 2\tanh\left[\frac{B}{10R}\left(\left(\frac{3B^2}{25R} - a_0\right)t + x\right)\right]\right), \quad (4.1.26)$$

where the arbitrary constant a_0 affects the solution (and therefore its boundary condition) as well as the velocity of the stationary wave. From (4.1.26) and Eqs. (4.1.2)- (4.1.3), we arrive at a set of exact stochastic solutions of Eq. (4.1.1), which are simplified as follows:

$$U(X,T) = a_0 - \frac{3B^2}{25R}(\tanh^2[\phi(X,T)] + 2\tanh[\phi(X,T)]) + W(T), \qquad (4.1.27)$$

or

$$U(X,T) = a_0 - \frac{3B^2}{25R}(1 - sech^2[\phi(X,T)] + 2\tanh[\phi(X,T)]) + W(T), \qquad (4.1.28)$$

where

$$\phi(X,T) = \left[\frac{B}{10R}\left(\left(\frac{3B^2}{25R} - a_0\right)T + X - \int_0^T W(T')dT'\right)\right]$$

Remark 1. In (4.1.27) if we take $a_0 = -\frac{3B^2}{25R}$ we obtain solution (4.1.14).

4.1.3. Visualization of Some Solutions

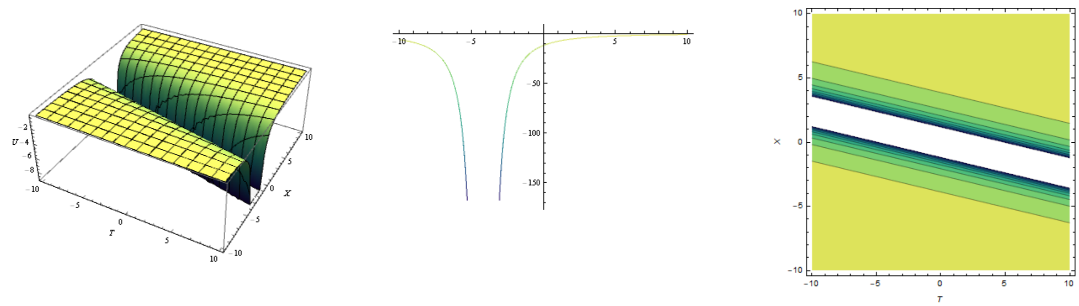

Figure 4.1. 3D, 2D, Contour Plots of the solution (4.1.15) for $B = R = 1$, where $W(T) = 0$.

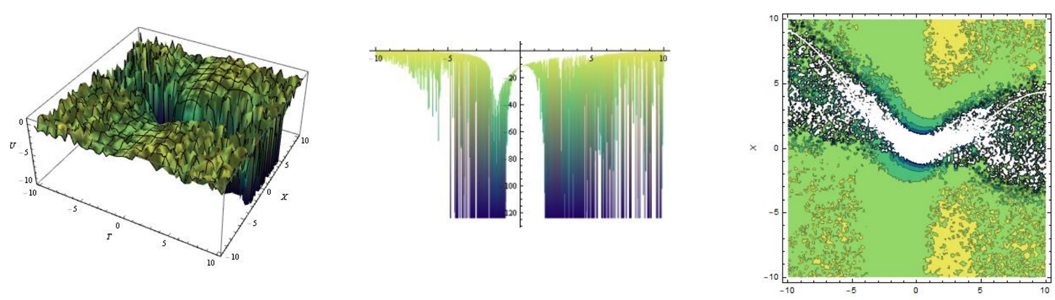

Figure 4.2. 3D, 2D, Contour Plots of the solution (4.1.15) for B=R=1, where $W(T) = \sin[noise * T]$.

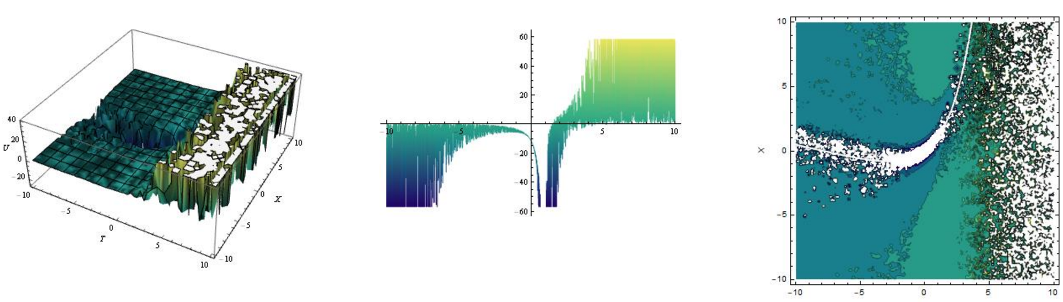

Figure 4.3. 3D, 2D, Contour Plots of the solution (4.1.15) for B=R=1, where $W(T) = exp(noise * T)$.

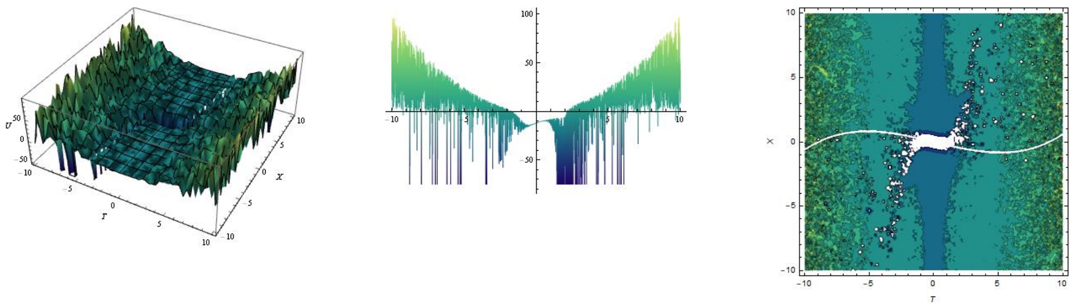

Figure 4.4. 3D, 2D, Contour Plots of the solution (4.1.15) for B=R=1, where $W(T) = noise * T^2$.

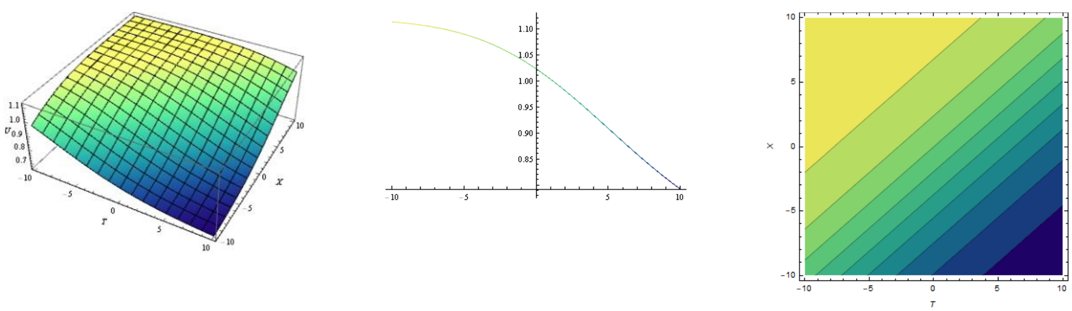

Figure 4.5. 3D, 2D, Contour Plots of the solution (4.1.28) for B=R=1, where $W(T) = 0$.

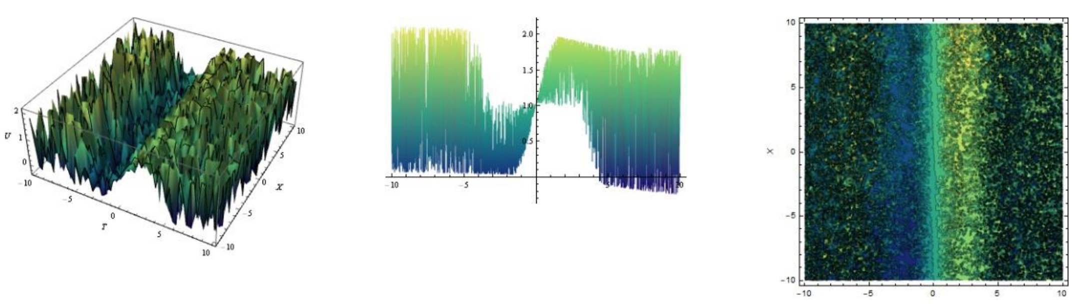

Figure 4.6. 3D, 2D, Contour Plots of the solution (4.1.28) for B=R=1, where $W(T) = \sin[noise * T]$.

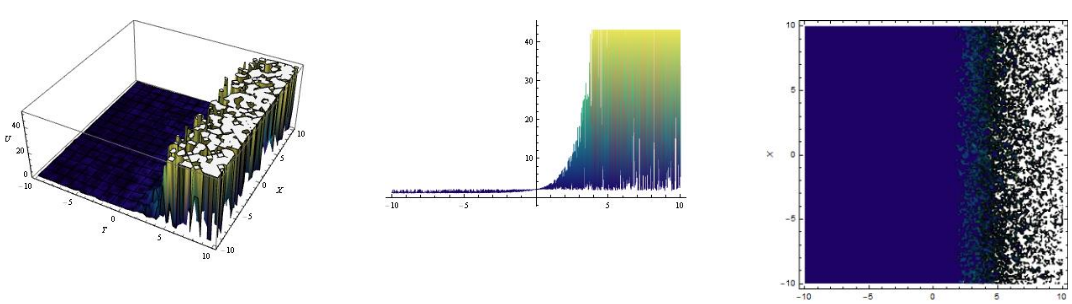

Figure 4.7. 3D, 2D, Contour Plots of the solution (4.1.28) for B=R=1, where $W(T) = exp(noise * T)$

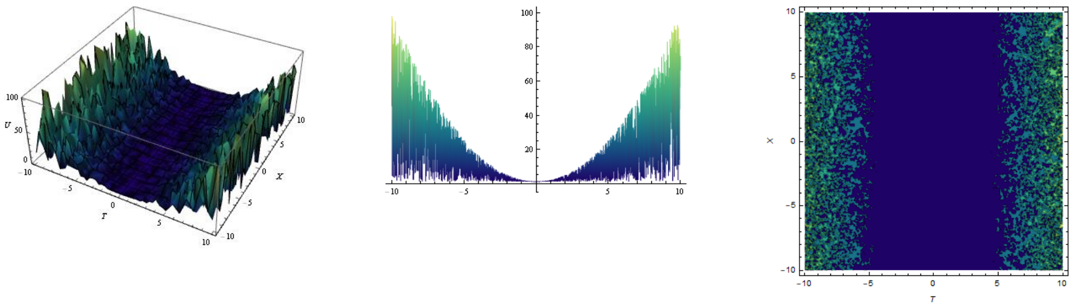

Figure 4.8. 3D, 2D, Contour Plots of the solution (4.1.28) for B=R=1, where $W(T) = noise * T^2$.

4.2. Using Galilean Transform for Solving the Stochastic KdV–Burgers Equation Via The Method of Extended Tanh

4.2.1. The method of extended tanh with zero boundary condition

Based on what was previously found, the balancing parameter takes the value $M = 2$. Hence using (3.2.9) the next step of the method of extended tanh gives the following finite expansion

$$V(z) = a_0 + a_1 Y + a_2 Y^2 + a_{-1} Y^{-1} + a_{-2} Y^{-2}, Y = tanh(z). \tag{4.2.1}$$

From the two relations (4.2.1) and (4.1.5), we get the following system of algebraic equations for $a_0, a_1, a_2, a_{-1}, a_{-2}, k$ and c after collecting the $Y^S, (S = 0,1, \dots ,8)$ coefficients with each other and equating them to zero

$$
\begin{aligned}
&Y^0 .: \quad \frac{1}{2} a_{-2}^2 + 6Rk^2 a_{-2} = 0,\\
&Y^1 .: \quad 2Rk^2 a_{-1} + 2Bka_{-2} + a_{-2} a_{-1} = 0\\
&Y^2 .: \quad \frac{1}{2} a_{-1}^2 - ca_{-2} + a_{-2} a_0 - 8Rk^2 a_{-2} + Bka_{-1} = 0\\
&Y^3 .: \quad -2Rk^2 a_{-1} - 2Bka_{-2} - ca_{-1} + a_{-2} a_1 + a_{-1} a_0 = 0.\\
&Y^4 .: \quad \frac{1}{2} a_0^2 - ca_0 + a_{-2} a_2 + a_{-1} a_1 - Bka_1 + 2Rk^2 a_{-2} + 2Rk^2 a_2 - Bka_{-1} = 0\\
&Y^5 .: \quad -2Rk^2 a_1 - 2Bka_2 - ca_1 + a_{-1} a_2 + a_0 a_1 = 0\\
&Y^6 .: \quad \frac{1}{2} a_1^2 + a_0 a_2 - 8Rk^2 a_2 + Bka_1 - ca_2 = 0\\
&Y^7 .: \quad 2Rk^2 a_1 + 2Bka_2 + a_1 a_2 = 0\\
&Y^8 .: \quad \frac{1}{2} a_2^2 + 6Rk^2 a_2 = 0 .
\end{aligned} \tag{4.2.2}
$$

Now, using one of the symbolic calculation programs to solve the above algebraic system, we get the following four cases of solutions:

1.

$$c=\frac{6B^2}{25R},\quad k=\frac{B}{10R},\quad a_{-2}=-\frac{3B^2}{25R},\quad a_{-1}=-\frac{6B^2}{25R},\quad a_0=\frac{9B^2}{25R},\quad a_1=0,\quad a_2=0. \tag{4.2.3}$$

2.

$$c=-\frac{6B^2}{25R},\quad k=\frac{B}{10R},\quad a_{-2}=-\frac{3B^2}{25R},\quad a_{-1}=-\frac{6B^2}{25R},\quad a_0=-\frac{3B^2}{25R},\quad a_1=0,\quad a_2=0. \tag{4.2.4}$$

3.

$$c=\frac{6B^2}{25R},k=\frac{B}{20R},a_{-2}=-\frac{3B^2}{100R},\quad a_{-1}=-\frac{3B^2}{25R},a_0=\frac{3B^2}{10R},a_1=-\frac{3B^2}{25R},a_2=-\frac{3B^2}{100R}. \tag{4.2.5}$$

4.

$$c=-\frac{6B^2}{25R},k=\frac{B}{20R},a_{-2}=-\frac{3B^2}{100R},a_{-1}=-\frac{3B^2}{25R},a_0=-\frac{9B^2}{50R},a_1=-\frac{3B^2}{25R},a_2=-\frac{3B^2}{100R}. \tag{4.2.6}$$

Solutions of the Case 1. Substituting the solutions (4.2.3) into (3.2.9) and using (3.2.6), (3.2.2), we obtain the following soliton solution;

$$u_1(x,t)=\frac{3B^2}{25R}\coth^2\left[\frac{B}{10R}\left(\frac{6B^2}{25R}t-x\right)\right]\left(\tanh\left[\frac{B}{10R}\left(\frac{6B^2}{25R}t-x\right)\right]+1\right)\left(3\tanh\left[\frac{B}{10R}\left(\frac{6B^2}{25R}t-x\right)\right]-1\right). \tag{4.2.7}$$

We should note that in the above solution if we take $\phi=\frac{B}{10R}\left(x-\frac{6B^2}{25R}t\right)$ then $\tanh\phi\to 1$ when $\phi\to\infty$, and the solution can be written as $u_1(x,t)=\frac{3B^2}{25R}(1-\tanh\phi)\cdots$ satisfies boundary conditions. From (4.2.7) and Eqs. (4.1.2)-(4.1.3), we arrive at a set of exact stochastic solutions of Eq. (4.1.1), which are simplified as follows:

$$U_1(X,T)=\frac{3B^2}{25R}\coth^2[\phi_1(X,T)]\,(\tanh[\phi_1(X,T)]+1)(3\tanh[\phi_1(X,T)]-1)+W(T), \tag{4.2.8}$$

where

$$\phi_1(X,T)=\left[\frac{B}{10R}\left(\frac{6B^2}{25R}T-X+\int_0^T W(T')\,dT'\right)\right]$$

Solutions of the Case 2. Substituting the solutions (4.2.4) into (3.2.9) and using (3.2.6), (3.2.2), we obtain the following soliton solution.

$$u_2(x,t) = -\frac{3B^2}{25R}\coth^2\left[\frac{B}{10R}\left(\frac{6B^2}{25R}t + x\right)\right]\left(\tanh\left[\frac{B}{10R}\left(\frac{6B^2}{25R}t + x\right)\right] + 1\right)^2. \quad (4.2.9)$$

We should note that in the above solution if we take $\phi = \frac{B}{10R}\left(\frac{6B^2}{25R}t + x\right)$ then $\tanh\phi \to -1$ when $\phi \to -\infty$, and the solution can be written as $u_2(x,t) = -\frac{3B^2}{25R}\coth^2[\phi]\,(1 + \tanh[\phi])^2$ satisfies boundary conditions. From (4.2.94.2.7) and Eqs. (4.1.2)-(4.1.3), we arrive at a set of exact stochastic solutions of Eq. (4.1.1), which are simplified as follows:

$$U_2(X,T) = -\frac{3B^2}{25R}\coth^2[\phi_2(X,T)]\,(\tanh[\phi_2(X,T)] + 1)^2 + W(T). \qquad (4.2.10)$$

Where

$$\phi_2(X,T) = \left[\frac{B}{10R}\left(\frac{6B^2}{25R}T + X - \int_0^T W(T')dT'\right)\right]$$

Solutions of the Case 3. Substituting the solutions (4.2.54.2.4) into (3.2.9) and using (3.2.6), (3.2.2), we obtain the following soliton solution.

$$u_3(x,t) = -\frac{3B^2}{100R}\coth^2[\phi_3(x,t)]\,(\tanh^2[\phi_3(x,t)] - 6\tanh[\phi_3(x,t)] + 1)(\tanh[\phi_3(x,t)] + 1)^2, \quad (4.2.11)$$

where

$$\phi_3(x,t) = \left[\frac{B}{20R}\left(\frac{6B^2}{25R}t - x\right)\right].$$

We should note that in the above solution if we take $\phi = \left[x - \frac{B}{20R}\left(\frac{6B^2}{25R}t\right)\right] = -\phi_3(x,t)$ then $\tanh\phi \to 1$ when $\phi \to \infty$, and the solution can be written as $u_3(x,t) = -\frac{3B^2}{100R}(1 - \tanh[\phi])^2$ satisfies boundary conditions. From (4.2.11) ,(4.2.9) ,(4.2.7) and Eqs. (4.1.2)-(4.1.3), we arrive at a set of exact stochastic solutions of Eq. (4.1.1), which are simplified as follows:

$$U_3(X,T) = -\frac{3B^2}{100R}\coth^2[\phi_3(X,T)]\,(\tanh^2[\phi_3(X,T)] - 6\tanh[\phi_3(X,T)] + 1)(\tanh[\phi_3(X,T)] + 1)^2 + W(T), \quad (4.2.12)$$

where $\phi_3(X,T) = \left[\frac{B}{20R}\left(\frac{6B^2}{25R}T - X + \int_0^T W(T')dT'\right)\right]$.

Solutions of the Case 3. Substituting the solutions (4.2.64.2.54.2.4) into (3.2.9) and using (3.2.6), (3.2.2), we obtain the following soliton solution.

$$u_4(x,t) = -\frac{3B^2}{100R}\coth^2\left[\frac{B}{20R}\left(\frac{6B^2}{25R}t+x\right)\right]\left(\tanh\left[\frac{B}{20R}\left(\frac{6B^2}{25R}t+x\right)\right]+1\right)^4. \quad (4.2.13)$$

We should note that in the above solution if we take $\phi = \frac{B}{20R}\left(\frac{6B^2}{25R}t+x\right)$ then $\tanh\phi \to -1$ when $\phi \to -\infty$, and the solution written as $u_4(x,t) = -\frac{3B^2}{100R}(1+\tanh[\phi])^4\coth^2[\phi]$ satisfies boundary conditions. From (4.2.13) ,(4.2.11) ,(4.2.9) ,(4.2.7) and Eqs. (4.1.2)-(4.1.3), we arrive at a set of exact stochastic solutions of Eq. (4.1.1), which are simplified as follows:

$$U_4(X,T) = -\frac{3B^2}{100R}\coth^2[\phi_4(X,T)]\,(\tanh[\phi_4(X,T)]+1)^4 + W(T), \qquad (4.2.14)$$

where

$$\phi_4(X,T) = \left[\frac{B}{20R}\left(\frac{6B^2}{25R}T + X - \int_0^T W(T')\,dT'\right)\right]$$

4.2.2. The method of extended tanh without boundary condition

Based on what was previously found, the balancing parameter takes the value $M = 2$. Hence, using (3.2.9) the next step of the method of extended tanh gives the following finite expansion

$$V(z) = a_0 + a_1Y + a_2Y^2 + a_{-1}Y^{-1} + a_{-2}Y^{-2}, Y = tanh(z). \qquad (4.2.15)$$

From the two relations (4.2.15) and (4.1.20), we get to the following system of algebraic equations for $a_0, a_1, a_2, a_{-1}, a_{-2}, k$ and c after collecting the Y^S, $(S = 0,1, \dots ,10)$ coefficients to each other and equating them to zero.

Y^0 .: $\quad -24Rk^2a_{-2} - 2a_{-2}^2 = 0,$

Y^1 .: $\quad -6Rk^2a_{-1} - 6Bka_{-2} - 3a_{-2}a_{-1} = 0$

Y^2 .: $\quad 40Rk^2a_{-2} - 2Bka_{-1} + 2ca_{-2} + 2a_{-2}^2 - 2a_{-2}a_0 - a_{-1}^2 = 0$

Y^3 .: $\quad 8Rk^2a_{-1} + 8Bka_{-2} + ca_{-1} + 3a_{-2}a_{-1} - a_{-2}a_1 - a_{-1}a_0 = 0$

Y^4 .: $\quad -16Rk^2a_{-2} + 2Bka_{-1} - 2ca_{-2} + 2a_{-2}a_0 + a_{-1}^2 = 0$

Y^5 .: $\quad -2Rk^2a_{-1} - 2Rk^2a_1 - 2Bka_{-2} - 2Bka_2 - ca_{-1} - ca_1 + a_{-2}a_1 + a_{-1}a_0 + a_{-1}a_2 + a_0a_1 = 0$

Y^6 .: $\quad -16Rk^2a_2 + 2Bka_1 - 2ca_2 + 2a_0a_2 + a_1^2 = 0,$ (4.2.16)

Y^7 .: $\quad 8Rk^2a_1 + 8Bka_2 + ca_1 - a_{-1}a_2 - a_0a_1 + 3a_1a_2 = 0$

Y^8 .: $\quad 40Rk^2a_2 - 2Bka_1 + 2ca_2 - 2a_0a_2 - a_1^2 + 2a_2^2 = 0$

Y^9 .: $\quad -6Rk^2a_1 - 6Bka_2 - 3a_1a_2 = 0$

Y^{10} .: $\quad -24Rk^2a_2 - 2a_2^2 = 0$

Now, using one of the symbolic calculation programs to solve the above algebraic system, we get the following solution:

$$c = a_0 - \frac{3B^2}{50R}, k = -\frac{B}{20R}, a_{-2} = -\frac{3B^2}{100R}, a_{-1} = \frac{3B^2}{25R}, a_0 = a_0, a_1 = \frac{3B^2}{25R}, a_2 = -\frac{3B^2}{100R}, \qquad (4.2.17)$$

Substituting the solutions (4.2.17) into (3.2.9) and using (3.2.6), (3.2.2), we obtain the following soliton solution.

$$u(x,t) = a_0 + \frac{6B^2}{50R}\left(\coth^2\left[\frac{B}{10R}\left(\left(a_0 - \frac{3B^2}{50R}\right)t - x\right)\right] + 2\coth\left[\frac{B}{10R}\left(\left(a_0 - \frac{3B^2}{50R}\right)t - x\right)\right] - \frac{1}{2}\right), \qquad (4.2.18)$$

where the arbitrary constant a_0 affects the solution (and therefore its boundary condition) as well. From (4.2.18) and Eqs. (4.1.2)-(4.1.3), we arrive at a set of exact stochastic solutions of Eq. (4.1.1), which are simplified as follows:

$$U(X,T) = a_0 + \frac{6B^2}{50R}\left(\coth^2[\phi(X,T)] + 2\coth[\phi(X,T)] - \frac{1}{2}\right) + W(T), \qquad (4.2.19)$$

where

$$\phi(X,T) = \left[\frac{B}{10R}\left(\left(a_0 - \frac{3B^2}{50R}\right)T - X + \int_0^T W(T')dT'\right)\right]$$

Remark 2. For cases (4.2.3) and (4.2.4) the obtained solutions are same with the solutions (4.1.17) and (4.1.15) respectively. But for cases (4.2.5) and (4.2.6) we obtain new solitary wave solutions.

4.2.3. Visualization of Some Solutions

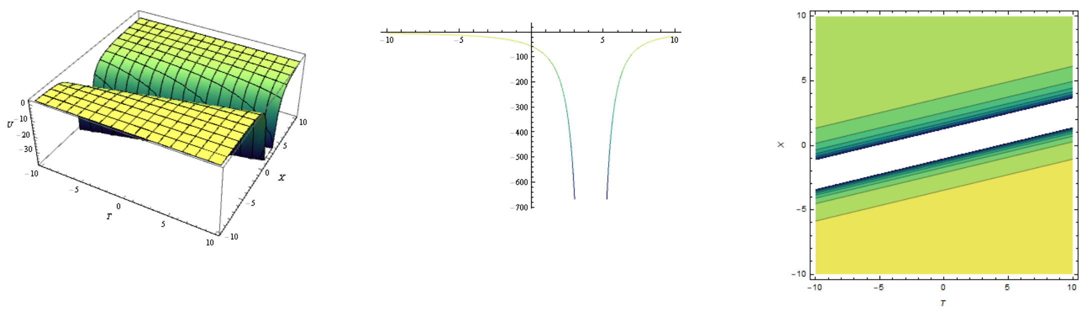

Figure 4.9. 3D, 2D, Contour Plots of the solution (4.2.12) for B=R=1, where $W(T) = 0$.

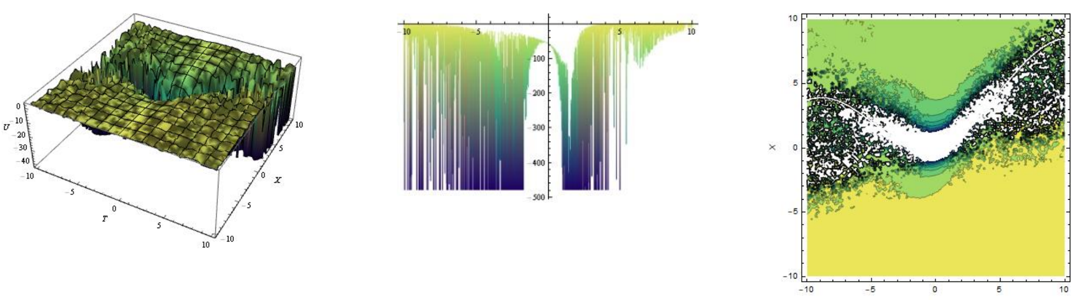

Figure 4.10. 3D, 2D, Contour Plots of the solution (4.2.12) for B=R=1, where $W(T) = \sin[noise * T]$.

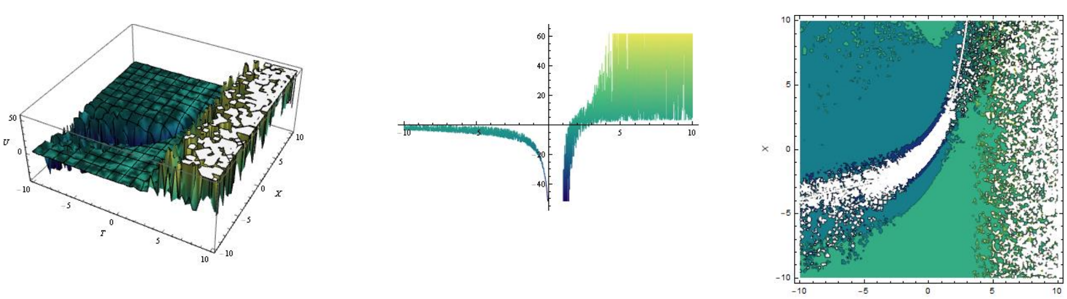

Figure 4.11. 3D, 2D, Contour Plots of the solution (4.2.12) for B=R=1, where $W(T) = exp(noise * T)$

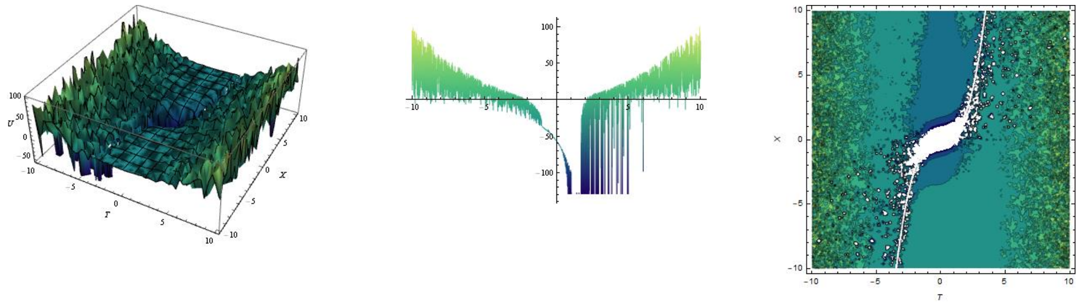

Figure 4.12. 3D, 2D, Contour Plots of the solution (4.2.12) for B=R=1, where $W(T) = noise * T^2$.

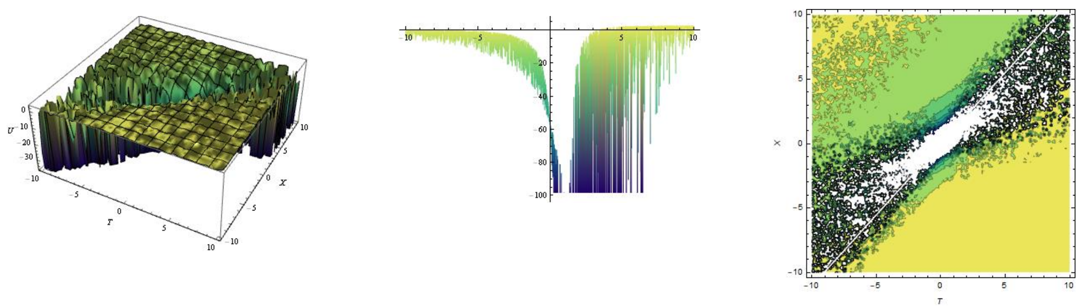

Figure 4.13. 3D, 2D, Contour Plots of the solution (4.2.12) for B=R=1, where $W(T) = noise * 1$.

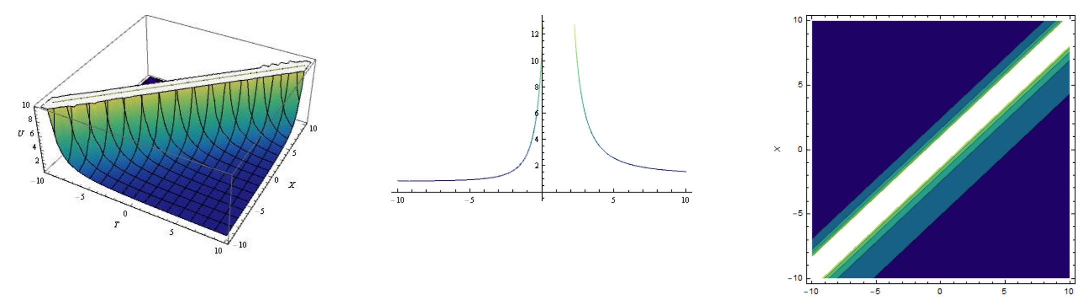

Figure 4.14. 3D, 2D, Contour Plots of the solution (4.2.19) for B=R=1, where $W(T) = 0$.

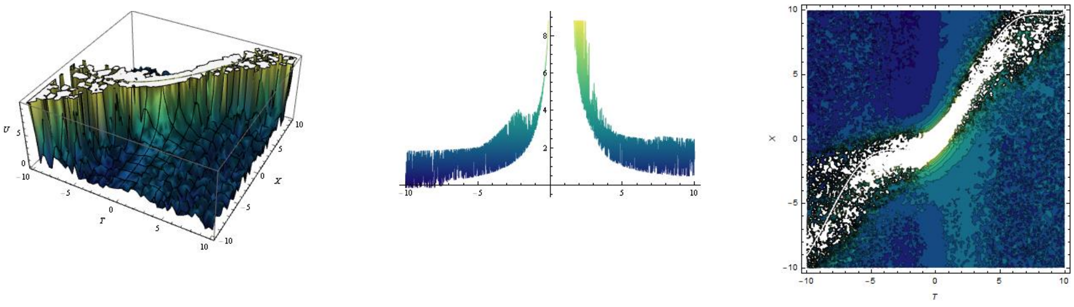

Figure 4.15. 3D, 2D, Contour Plots of the solution (4.2.19) for B=R=1, where $W(T) = \sin[noise * T]$.

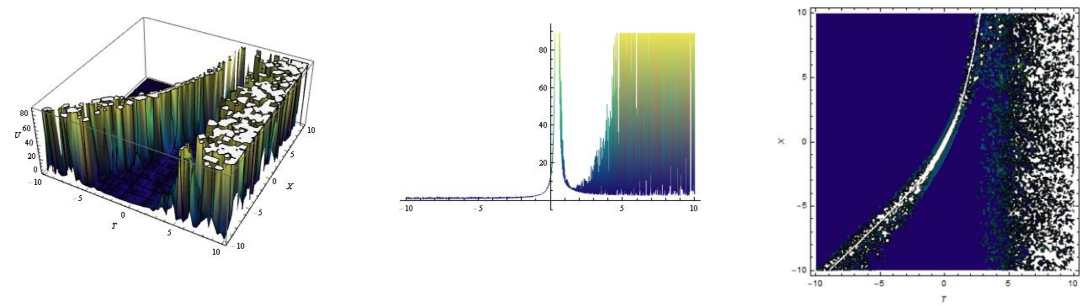

Figure 4.16. 3D, 2D, Contour Plots of the solution (4.2.19) for B=R=1, where $W(T) = exp(noise * T)$.

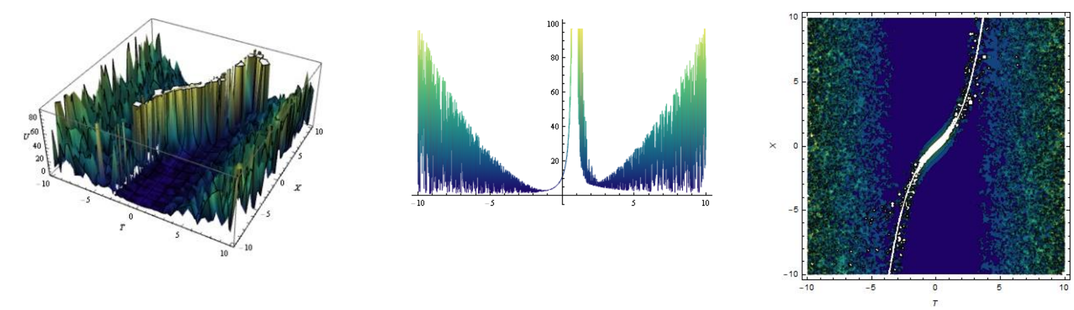

Figure 4.17. 3D, 2D, Contour Plots of the solution (4.2.19) for B=R=1, where $W(T) = noise * T^2$.

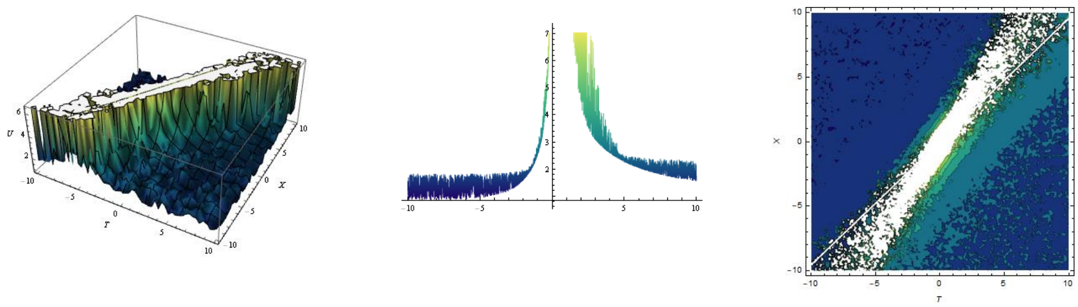

Figure 4.18. 3D, 2D, Contour Plots of the solution (4.2.19) for B=R=1, where $W(T) = noise * 1$.

4.3. Using Galilean Transform for Solving the Stochastic Korteweg–De Vries (KdV) Equation Via the Method of Tanh

The KdV equation is often used in the description of wave processes in dissipative-dispersive systems in many areas or physics (Ablowitz and Clarkson, 1991). Let us start from the following equation

$$U_t + UU_X + RU_{XXX} = \eta(T), \tag{4.3.1}$$

we call Eq. (4.3.1) a nonlinear stochastic Kortewegde Vries (KdV) equation. Here inhomogeneous term $\eta(T)$ stands for external noise and subscripts X and T denote partial differentiation with respect to X and T, respectively. One simply applies the Galilean transformation

$$U(X,T) = u(x,t) + W(T), x = X + m(t), t = T, \tag{4.3.2}$$

$$m(T) = -\int_0^T W(T')dT', \quad W(T) = \int_0^T \eta(T')dT', \tag{4.3.3}$$

to transform the stochastic Kortewegde Vries into Kortewegde Vries equation

$$u_t + uu_x + Ru_{xxx} = 0. \tag{4.3.4}$$

In the following sections, we will first use the tanh method to develop solitary wave solutions to the Stochastic Kortewegde Vries equation (4.3.1). The extended tanh method will be employed as well to develop more new solitary wave solutions.

4.3.1. The method of tanh with zero boundary condition

Introducing the following wave variable $u(x,t) = V(z), z = k(x - ct)$ carries the Korteweg de Vries equation (4.3.4) into an ODE "*with zero boundary condition*"

$$-ckV + \frac{1}{2}kV^2 + Rk^3V'' = 0. \tag{4.3.5}$$

Balancing $V'' with\ V^2$in (4.1.20) gives

$$M + 2 = 2M \tag{4.3.6}$$

Therefore

$$M = 2 \tag{4.3.7}$$

Hence the next step of the method of tanh gives the following finite expansion

$$V(z) = a_0 + a_1Y + a_2Y^2, Y = tanh(z) \tag{4.3.8}$$

From the two relations (4.3.8) and (4.3.5), we get to the following system of algebraic equations for a_0, a_1, a_2, k and c after collecting the $Y^S, (S = 0,1, \ldots, 4)$ coefficients to each other and equating them with zero

$$\begin{aligned}
&Y^0 .: \quad \tfrac{1}{2}ka_0^2 + 2a_2Rk^3 - cka_0 = 0\\
&Y^1 .: \quad -2Rk^3a_1 - cka_1 + ka_0a_1 = 0\\
&Y^2 .: \quad ka_0a_2 + \frac{1}{2}ka_1^2 - 8a_2Rk^3 - cka_2 = 0.\\
&Y^3 .: \quad 2Rk^3a_1 + ka_1a_2 = 0\\
&Y^4 .: \quad \tfrac{1}{2}ka_2^2 + 6a_2Rk^3 = 0
\end{aligned} \tag{4.3.9}$$

Now, using one of the symbolic calculation programs to solve the above algebraic system, we get the following solution:

$$c = 4k^2R, \quad k = k, \quad a_0 = 12k^2R, \quad a_1 = 0, \quad a_2 = -12k^2R. \tag{4.3.10}$$

Substituting the solutions (4.3.10) into (4.3.8) and using (3.2.6),(3.2.2), we arrive to the following solutions;

$$u(x,t) = 12k^2R(1-\tanh[k(4Rk^2t-x)])(1+\tanh[k(4Rk^2t-x)]), \qquad (4.3.11)$$

or

$$u(x,t) = 12k^2R(1-\tanh^2[k(4Rk^2t-x)]) = 12k^2Rsech^2[k(4Rk^2t-x)], \quad (4.3.12)$$

the well-known solitary wave in bell-shape form. We should note that in the above solution if we take $\phi = k(4Rk^2t-x)$ then $\tanh\phi \to 1$ when $\phi \to \infty$, and the solution satisfies boundary conditions. From (4.3.11) - (4.3.12) and Eqs. (4.3.2) - (4.3.3), we arrive at a set of exact stochastic solutions of Eq. (4.3.1), which are simplified as follows:

$$U(X,T) = 12k^2R(1-\tanh[\phi(X,T)])(1+\tanh[\phi(X,T)]) + W(T), \qquad (4.3.13)$$

or

$$\begin{aligned} U(X,T) &= 12k^2R(1-\tanh^2[\phi(X,T)]) + W(T) \\ &= 12k^2R\,\mathrm{sech}^2[\phi(X,T)] + W(T) \end{aligned}, \qquad (4.3.14)$$

where

$$\phi(X,T) = \left[k\left(4Rk^2t - X + \int_0^T W(T')dT'\right)\right]$$

4.3.2. The method of tanh without boundary condition

Introducing the following wave variable $u(x,t) = V(z), z = k(x-ct)$ carries the Korteweg de Vries equation (4.3.4) into an ODE with "*no boundary conditions*"

$$-ckV' + kVV' + Rk^3V''' = 0. \qquad (4.3.15)$$

Balancing V''' with $V'V$ in (4.1.20) gives

$$M+3 = M+1+M. \qquad (4.3.16)$$

So that

$$M = 2. \qquad (4.3.17)$$

Hence the next step of the method of tanh gives the following finite expansion

$$V(z) = a_0 + a_1 Y + a_2 Y^2, Y = tanh(z). \tag{4.3.18}$$

From the two relations (4.3.18) and (4.3.15), we get to the following system of algebraic equations for a_0, a_1, a_2, k and c after collecting the $Y^S, (S = 0,1, \dots ,5)$ coefficients with each other and equating them to zero

$$
\begin{aligned}
Y^0 &.: \quad -2Rk^3 a_1 - cka_1 + ka_0 a_1 = 0 \\
Y^1 &.: \quad -16Rk^3 a_2 - 2cka_2 + 2ka_0 a_2 + ka_1^2 = 0 \\
Y^2 &.: \quad 8Rk^3 a_1 + cka_1 - ka_0 a_1 + 3ka_1 a_2 = 0 \\
Y^3 &.: \quad 40Rk^3 a_2 + 2cka_2 - 2ka_0 a_2 - ka_1^2 + 2ka_2^2 = 0. \qquad (4.3.19) \\
Y^4 &.: \quad -6Rk^3 a_1 - 3ka_1 a_2 = 0 \\
Y^5 &.: \quad -24Rk^3 a_2 - 2ka_2^2 = 0
\end{aligned}
$$

Now, using one of the symbolic calculation programs to solve the above algebraic system, we get the following solution:

$$c = a_0 - 8Rk^2, \quad k = k, \quad a_0 = a_0, \quad a_1 = 0, \quad a_2 = -12Rk^2. \tag{4.3.20}$$

Substituting the solutions (4.3.20) into (4.3.18) and using (3.2.6), (3.2.2), we arrive to the following solutions;

$$u(x,t) = a_0 - 12Rk^2 \tanh^2\left[k\left((8Rk^2 - a_0)t + x\right)\right], \tag{4.3.21}$$

and

$$u(x,t) = a_0 - 12Rk^2 \coth^2\left[k\left((8Rk^2 - a_0)t + x\right)\right], \tag{4.3.22}$$

where the arbitrary constant a_0 affects the solution (and therefore its boundary condition) as well as the velocity of the stationary wave. From (4.3.21)-(4.3.11) and Eqs. (4.3.2) - (4.3.3), we arrive at a set of exact stochastic solutions of Eq. (4.3.1), which are simplified as follows:

$$U(X,T) = a_0 - 12Rk^2 \tanh^2[\phi(X,T)] + W(T), \tag{4.3.23}$$

and

$$U(X,T) = a_0 - 12Rk^2 \coth^2[\phi(X,T)] + W(T), \tag{4.3.24}$$

or

$$U(X,T) = a_0 - 12Rk^2 + 12Rk^2 sech^2[\phi(X,T)] + W(T), \tag{4.3.25}$$

and

$$U(X,T) = a_0 - 12Rk^2 - 12Rk^2 csch^2[\phi(X,T)] + W(T), \tag{4.3.26}$$

where

$$\phi(X,T) = \left[\left((8Rk^2 - a_0)T + X - \int_0^T W(T')dT'\right)\right].$$

Remark 3. In (4.3.25) if we take $a_0 = 12Rk^2$ we obtain solution (4.3.14).

4.3.3. Visualization of Some Solutions

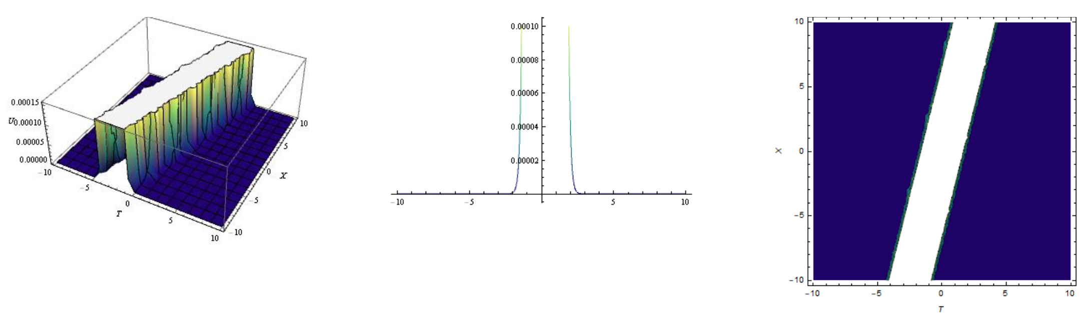

Figure 4.19. 3D, 2D, Contour Plots of the solution (4.3.14) for $B = R = 1$, where $W(T) = 0$.

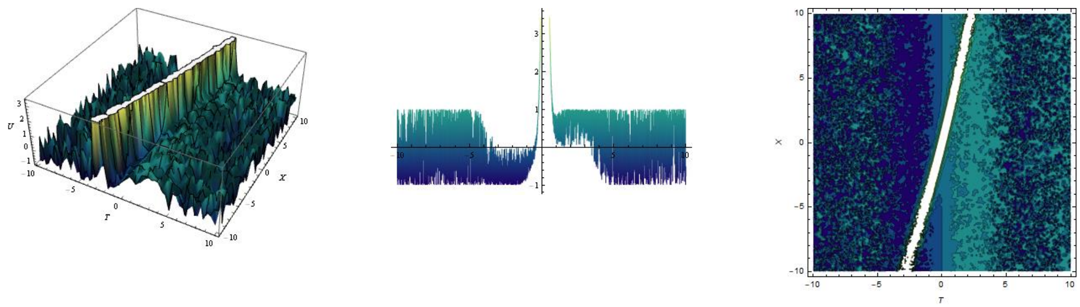

Figure 4.20. 3D, 2D, Contour Plots of the solution (4.3.14) for B=R=1, where $W(T) = \sin[noise * T]$.

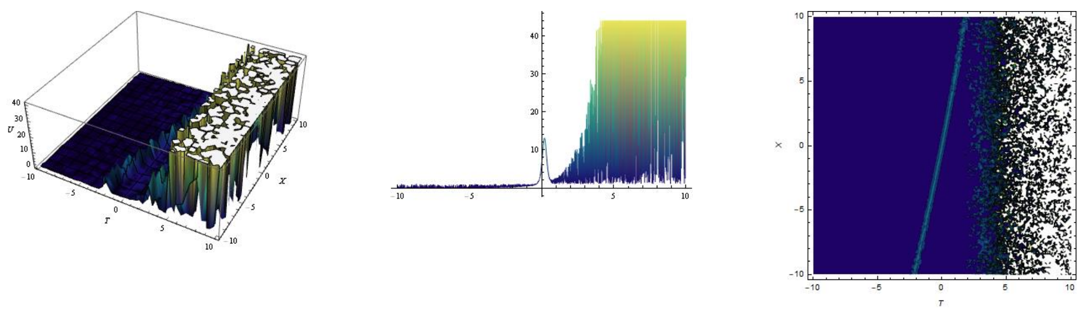

Figure 4.21. 3D, 2D, Contour Plots of the solution (4.3.14) for B=R=1, where $W(T) = exp(noise * T)$.

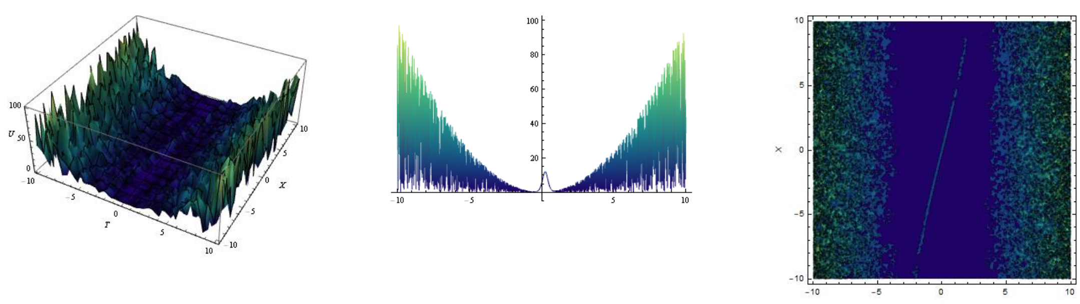

Figure 4.22. 3D, 2D, Contour Plots of the solution (4.3.14) for B=R=1, where $W(T) = noise * T^2$.

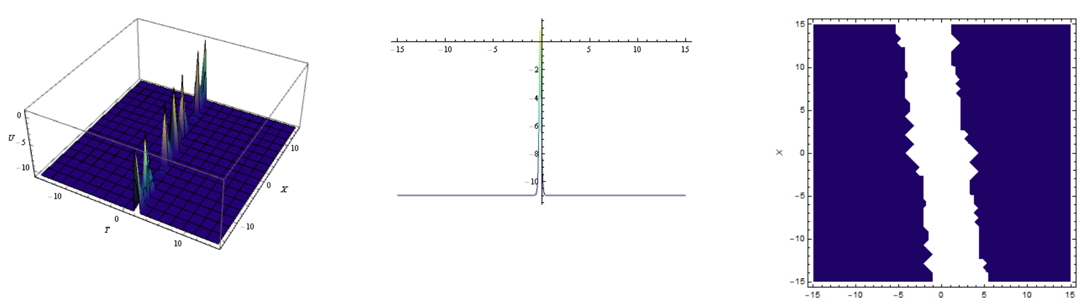

Figure 4.23. 3D, 2D, Contour Plots of the solution (4.3.23) for B=R=1, where $W(T) = 0$.

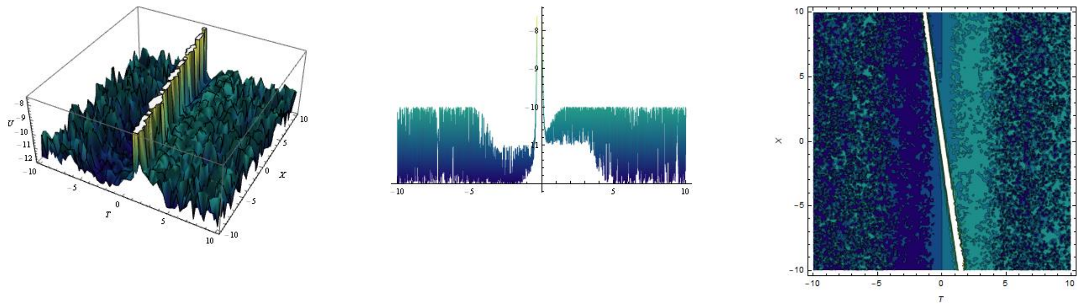

Figure 4.24. 3D, 2D, Contour Plots of the solution (4.3.23) for B=R=1, where $W(T) = \sin[noise * T]$.

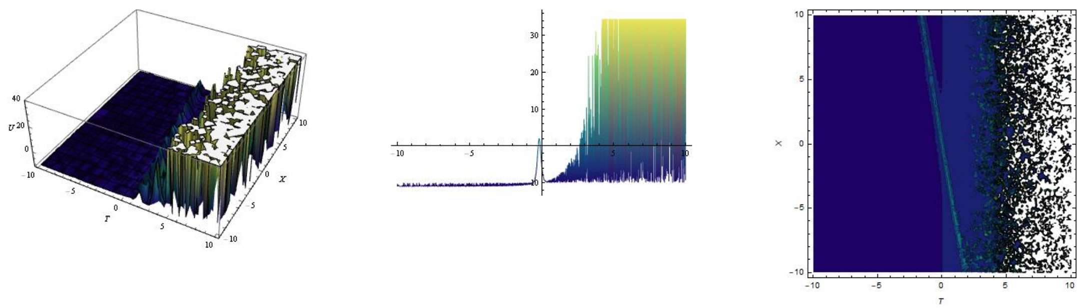

Figure 4.25. 3D, 2D, Contour Plots of the solution (4.3.23) for B=R=1, where $W(T) = exp(noise * T)$.

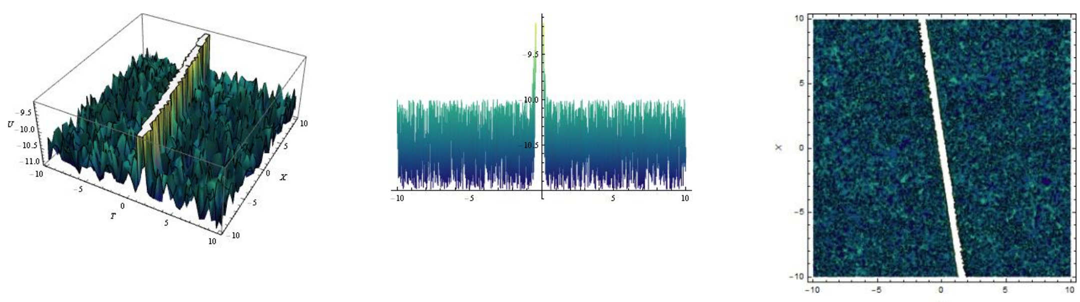

Figure 4.26. 3D, 2D, Contour Plots of the solution (4.3.23) for B=R=1, where $W(T) = noise * 1$.

4.4. Using Galilean Transform for Solving the Stochastic Korteweg–De Vries (KdV) Equation Via the Method of Extended Tanh

4.4.1. The method of extended tanh with zero boundary condition

Based on what was previously found, the balancing parameter takes the value $M = 2$. Hence using (3.2.9) the next step of the method of extended tanh gives the following finite expansion

$$V(z) = a_0 + a_1 Y + a_2 Y^2 + a_{-1} Y^{-1} + a_{-2} Y^{-2}, Y = tanh(z). \qquad (4.4.1)$$

From the two relations (4.4.1) and (4.3.5), we get to the following system of algebraic equations for $a_0, a_1, a_2, a_{-1}, a_{-2}, k$ and c after collecting the Y^S, $(S = 0,1, \ldots ,8)$ coefficients to each other and equating them with zero

Y^0 .: $\quad 6Rk^2 a_{-2} + \frac{1}{2} a_{-2}^2 = 0$

Y^1 .: $\quad 2Rk^2 a_{-1} + a_{-2} a_{-1} = 0$

Y^2 .: $\quad \frac{1}{2} a_{-1}^2 - c a_{-2} + a_{-2} a_0 - 8Rk^2 a_{-2} = 0$

Y^3 .: $\quad -2Rk^2 a_{-1} - c a_{-1} + a_{-2} a_1 + a_{-1} a_0 = 0$

Y^4 .: $\quad \frac{1}{2} a_0^2 - c a_0 + a_{-2} a_2 + a_{-1} a_1 + 2Rk^2 a_{-2} + 2Rk^2 a_2 = 0. \qquad (4.4.2)$

Y^5 .: $\quad -2Rk^2 a_1 - c a_1 + a_{-1} a_2 + a_0 a_1 = 0$

Y^6 .: $\quad \frac{1}{2} a_1^2 - c a_2 + a_0 a_2 - 8Rk^2 a_2 = 0$

Y^7 .: $\quad 2Rk^2 a_1 + a_1 a_2 = 0$

Y^8 .: $\quad \frac{1}{2} a_2^2 + 6Rk^2 a_2 = 0$

Now, using one of the symbolic calculation programs to solve the above algebraic system, we get the following cases of solutions:

1.
$$c = 4Rk^2, k = k, a_{-2} = -12Rk^2, a_{-1} = 0, a_0 = 12Rk^2, a_1 = 0, a_2 = 0. \qquad (4.4.3)$$

2.
$$c = 16Rk^2, k = k, a_{-2} = -12Rk^2, a_{-1} = 0, a_0 = 24Rk^2, a_1 = 0, a_2 = -12Rk^2. \qquad (4.4.4)$$

Solutions of the case 1. Substituting the solutions (4.4.3) into (3.2.9) and using (3.2.6), (3.2.2), we obtain the following soliton solution.

$$u_1(x,t) = -12Rk^2 coth^2[k(4Rk^2 t - x)]\, sech^2[k(4Rk^2 t - x)]. \qquad (4.4.5)$$

We should note that in the above solution if we take $\phi = k(4Rk^2t - x)$ and $sech^2[\phi] = 1 - \tanh^2[\phi]$ then $\tanh\phi \to 1$ $when\ \phi \to \infty$, and the solution satisfies boundary conditions. From (4.4.5)-(4.3.11) and Eqs. (4.3.2) - (4.3.3), we arrive at a set of exact stochastic solutions of Eq. (4.3.1), which are simplified as follows:

$$U_1(X,T) = -12Rk^2coth^2[\phi_1(X,T)]\, sech^2[\phi_1(X,T)] + W(T), \qquad (4.4.6)$$

where

$$\phi_1(X,T) = \left[k\left(4Rk^2T - X + \int_0^T W(T')dT'\right)\right]$$

Solutions of the Case 2. Substituting the solutions (4.4.4) into (3.2.9) and using (3.2.6), (3.2.2), we obtain the following soliton solution.

$$u_2(x,t) == -12Rk^2coth^2[k(16Rk^2t - x)]\, sech^4[k(16Rk^2t - x)]. \qquad (4.4.7)$$

We should note that in the above solution if we take $\phi = k(16Rk^2t - x)$ and $sech^2[\phi] = 1 - \tanh^2[\phi]$ then $\tanh\phi \to 1$ when $\phi \to \infty$, and the solution satisfies boundary conditions. From (4.4.5),(4.3.11) and Eqs. (4.3.2) - (4.3.3), we arrive at a set of exact stochastic solutions of Eq. (4.3.1), which are simplified as follows:

$$U_2(X,T) = -12Rk^2coth^2[\phi_1(X,T)]\, sech^4[\phi_1(X,T)] + W(T). \qquad (4.4.8)$$

where

$$\phi_2(X,T) = \left[k\left(16Rk^2T - X + \int_0^T W(T')dT'\right)\right]$$

4.4.2. The method of extended tanh without boundary condition

Using (3.2.9), the next step of the method of extended tanh gives the following finite expansion

$$V(z) = a_0 + a_1Y + a_2Y^2 + a_{-1}Y^{-1} + a_{-2}Y^{-2}, \qquad Y = tanh(z). \qquad (4.4.9)$$

From the two relations (4.4.9) and (4.3.15), we get to the following system of algebraic equations for $a_0, a_1, a_2, a_{-1}, a_{-2}, k$ and c after collecting the Y^S, $(S = 0,1, \dots, 10)$ coefficients to each other and equating them with zero

$$Y^0 .: \quad -24Rk^2a_{-2} - 2a_{-2}^2 = 0$$

$$Y^1 .: \quad -6Rk^2a_{-1} - 3a_{-2}a_{-1} = 0$$

$$Y^2 .: \quad 40Rk^2a_{-2} + 2ca_{-2} + 2a_{-2}^2 - 2a_{-2}a_0 - a_{-1}^2 = 0$$

$$Y^3 .: \quad 8Rk^2a_{-1} + ca_{-1} + 3a_{-2}a_{-1} - a_{-2}a_1 - a_{-1}a_0 = 0. \tag{4.4.10}$$

$$Y^4 .: \quad -16Rk^2a_{-2} - 2ca_{-2} + 2a_{-2}a_0 + a_{-1}^2 = 0$$

$$Y^5 .: \quad -2Rk^2a_{-1} - 2Rk^2a_1 - ca_{-1} - ca_1 + a_{-2}a_1 + a_{-1}a_0 + a_{-1}a_2 + a_0a_1 = 0$$

$$Y^6 .: \quad -16Rk^2a_2 - 2ca_2 + 2a_0a_2 + a_1^2 = 0$$

$$Y^7 .: \quad 8Rk^2a_1 + ca_1 - a_{-1}a_2 - a_0a_1 + 3a_1a_2$$

$$Y^8 .: \quad 40Rk^2a_2 + 2ca_2 - 2a_0a_2 - a_1^2 + 2a_2^2 = 0$$

$$Y^9 .: \quad -6Rk^2a_1 - 3a_1a_2 = 0$$

$$Y^{10} .: \quad -24Rk^2a_2 - 2a_2^2 = 0$$

Now, using one of the symbolic calculation programs to solve the above algebraic system, we get the following cases of solutions:

$$c = a_0 - 8Rk^2, \quad k = k, \quad a_{-2} = -12Rk^2, \quad a_{-1} = 0, \quad a_0 = a_0, \quad a_1 = 0, \quad a_2 = -12Rk^2. \tag{4.4.11}$$

Substituting the solutions (4.4.11) into (3.2.9) and using (3.2.6), (3.2.2), we arrive to the following solitons solution;

$$u(x,t) = a_0 - 12Rk^2\left(\tanh^2\left[k\left((a_0 - 8Rk^2)t + x\right)\right] + \coth^2\left[k\left((a_0 - 8Rk^2)t + x\right)\right]\right), \tag{4.4.12}$$

or

$$u(x,t) = a_0 + 24Rk^2 - 48Rk^2\coth^2\left[2k\left((a_0 - 8Rk^2)t + x\right)\right], \tag{4.4.13}$$

or

$$u(x,t) = a_0 - 24Rk^2 - 48Rk^2 csch^2\big[2k\big((a_0 - 8Rk^2)t + x\big)\big], \qquad (4.4.14)$$

where the arbitrary constant a_0 affects the solution. From (4.4.12)-(4.4.14) and Eqs. (4.3.2) - (4.3.3), we arrive at a set of exact stochastic solutions of Eq. (4.3.1), which are simplified as follows:

$$U(X,T) = a_0 - 12Rk^2(\tanh^2[\phi(X,T)] + \coth^2[\phi(X,T)]) + W(T), \qquad (4.4.15)$$

or

$$U(X,T) = a_0 + 24Rk^2 - 48Rk^2 \coth^2[2\phi(X,T)] + W(T), \qquad (4.4.16)$$

or

$$U(X,T) = a_0 - 24Rk^2 - 48Rk^2 csch^2[2\phi(X,T)] + W(T), \qquad (4.4.17)$$

where

$$\phi(X,T) = \left[k\left((a_0 - 8Rk^2)T + X - \int_0^T W(T')\right)\right]$$

Remark 4. Solutions (4.4.17) and (4.3.25) can be obtained from one to the other.

4.4.3. Visualization of Some Solutions

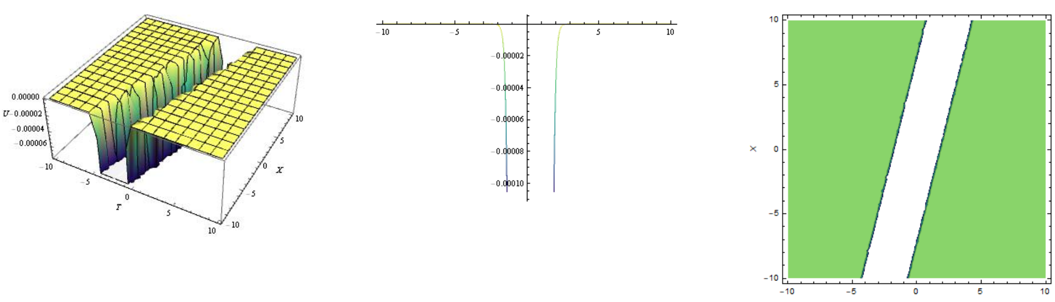

Figure 4.27. 3D, 2D, Contour Plots of the solution (4.4.6) for B=R=1, where $W(T) = 0$.

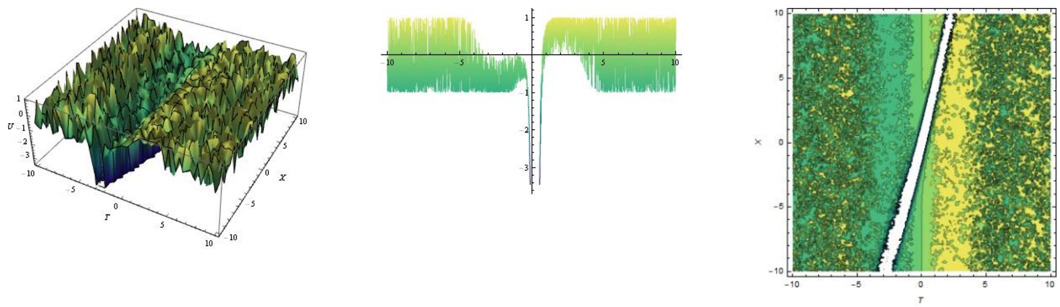

Figure 4.28. 3D, 2D, Contour Plots of the solution (4.4.6) for B=R=1, where $W(T) = \sin[noise * T]$.

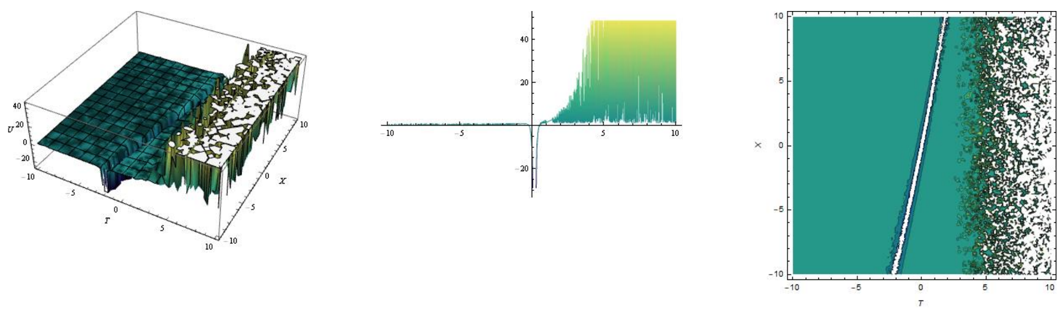

Figure 4.29. 3D, 2D, Contour Plots of the solution (4.4.6) for B=R=1,where $W(T) = exp(noise * T)$.

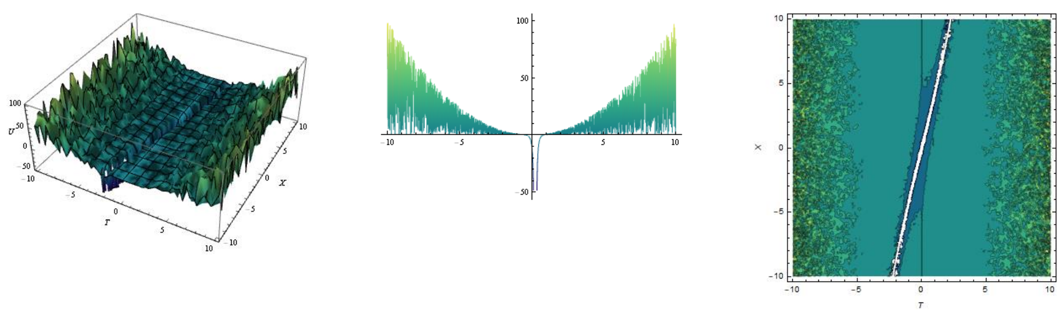

Figure 4.30. 3D, 2D, Contour Plots of the solution (4.4.6) for B=R=1, where $W(T) = noise * T^2$.

4.5. Using Galilean Transform for Solving the Stochastic Burgers' Equation Via the Method of Tanh

This equation, quoted as the most simple nonlinear wave equation, models fluid turbulence in a channel (Burgers, 1974). Also unidirectional sound waves in a gas, governed by the Navier Stokes equation, are described by such an equation (Karpman, 1975). The most appealing application, however, is its relation to shock waves in real fluids. Let us start from the following equation

$$U_t + UU_X - BU_{XX} = \eta(T). \qquad (4.5.1)$$

We call Eq.(4.5.1) a Burgers' nonlinear stochastic evolution equation.Here inhomogeneous term $\eta(T)$ stands for external noise and subscripts X and T denote partial differentiations with respect to X and T , respectively. One simply applies the Galilean transformation

$$U(X,T) = u(x,t) + W(T), \;\; x = X + m(t), t = T, \qquad (4.5.2)$$

$$m(T) = -\int_0^T W(T')dT', \quad W(T) = \int_0^T \eta(T')dT', \qquad (4.5.3)$$

to transform the stochastic Burgers' into deterministic Burgers' equation

$$u_t + uu_x - Bu_{xx} = 0. \qquad (4.5.4)$$

In the following sections, we will first use the tanh method to develop solitary wave solutions to the Stochastic Burgers' equation (4.5.1).

4.5.1. The method of tanh with zero boundary condition

Introducing the following wave variable $u(x,t) = V(z), z = k(x - ct)$ carries the Burgers' equation (4.5.4) into an ODE "*with zero boundary condition*"

$$-ckV + \frac{1}{2}kV^2 - Bk^2V' = 0. \qquad (4.5.5)$$

Balancing $V' with\, V^2$ in (4.5.5) gives

$$2M = M + 1. \qquad (4.5.6)$$

Therefore

$$M = 1. \qquad (4.5.7)$$

Hence the next step of the method of tanh gives the following finite expansion

$$V(z) = a_0 + a_1Y, Y = tanh(z). \qquad (4.5.8)$$

From the two relations (4.5.8) and (4.5.5), we get to the following system of algebraic equations for a_0, a_1, k and c after collecting the $Y^S, (S = 0,1, \dots ,2)$ coefficients to each other and equating them with zero

Y^0 .: $\frac{1}{2}ka_0^2 - Bk^2a_1 - cka_0 = 0$

$$Y^1 .: \quad -cka_1 + ka_0 a_1 = 0. \tag{4.5.9}$$

$$Y^2 .: \quad \frac{1}{2} k a_1^2 + Bk^2 a_1 = 0$$

Now, using one of the symbolic calculation programs to solve the above algebraic system, we get the following solution:

$$c = 2kB, \quad k = k, \quad a_0 = 2kB, \quad a_1 = -2kB \tag{4.5.10}$$

Substituting the solutions (4.5.10) into (4.5.8) and using (3.2.6), (3.2.2), we arrive to the following solitons solution

$$u(x,t) = 2kB(\tanh[k(2Bkt - x)] + 1) = -2kB(1 - \tanh[k(x - 2Bkt)]), \tag{4.5.11}$$

where the arbitrary constant a_0 affects the solution (and therefore its boundary condition) as well as the velocity of the stationary wave. From (4.5.11) and Eqs. (4.5.2)-(4.5.3), we arrive at a set of exact stochastic solutions of Eq. (4.5.1), which are simplified as follows:

$$U(X,T) = 2kB(\tanh[\phi(X,T)] + 1) + W(T), \tag{4.5.12}$$

where

$$\phi(X,T) = \left[k\left(2BkT - X + \int_0^T W(T')dT'\right)\right]$$

4.5.2. The method of tanh without boundary condition

Introducing the following wave variable $u(x,t) = V(z), z = k(x - ct)$ carries the Burgers' equation (4.5.4) into an ODE "*No boundary conditions*"

$$-ckV' + kVV' - Bk^2 V'' = 0. \tag{4.5.13}$$

Balancing V'' with $V'V$ in (4.5.13) gives

$$M + 2 = M + 1 + M. \tag{4.5.14}$$

Therefore

$$M = 1. \tag{4.5.15}$$

Hence the next step of the method of tanh gives the following finite expansion

$$V(z) = a_0 + a_1 Y, Y = tanh(z). \tag{4.5.16}$$

From the two relations (4.5.16) and (4.5.13), we get to the following system of algebraic equations for a_0, a_1, k and c after collecting the $Y^S, (S = 0,1, \dots ,3)$ coefficients to each other and equating them with zero

$$\begin{aligned}
&Y^0 .: \quad -cka_1 + ka_0a_1 = 0 \\
&Y^1 .: \quad 2Bk^2a_1 + ka_1^2 = 0. \\
&Y^2 .: \quad cka_1 - ka_0a_1 = 0 \\
&Y^3 .: \quad -2Bk^2a_1 - ka_1^2 = 0
\end{aligned} \tag{4.5.17}$$

Now, using one of the symbolic calculation programs such as Maple and mathematica to solve the above algebraic system, we get the following two cases of solutions:

$$c = a_0, \quad k = k, \quad a_0 = a_0, \quad a_1 = -2Bk. \tag{4.5.18}$$

Substituting the solutions (4.5.18) into (4.5.16) and using (3.2.6), (3.2.2), we arrive to the following solitons solution

$$u(x,t) = a_0 - 2Bk \tanh[k(x - a_0t)] , \tag{4.5.19}$$

and

$$u(x,t) = a_0 - 2Bk \coth[k(x - a_0t)] , \tag{4.5.20}$$

where the arbitrary constant a_0 affects the solution (and therefore its boundary condition) as well as the velocity of the stationary wave. From (4.5.19) and Eqs. (4.5.2)-(4.5.3), we arrive at a set of exact stochastic solutions of Eq. (4.5.1), which are in the form of a shock wave and simplified as follows:

$$U(X,T) = a_0 - 2Bk \tanh[\phi(X,T)] + W(T), \tag{4.5.21}$$

and

$$U(X,T) = a_0 - 2Bk \coth[\phi(X,T)] + W(T), \tag{4.5.22}$$

where

$$\phi(X,T) = \left[k\left(X - \int_0^T W(T')\,dT' - a_0 T\right)\right].$$

Remark 5. In (4.5.21) if a is chosen appropriately we obtain solution (4.5.12).

4.5.3. Visualization of some solutions

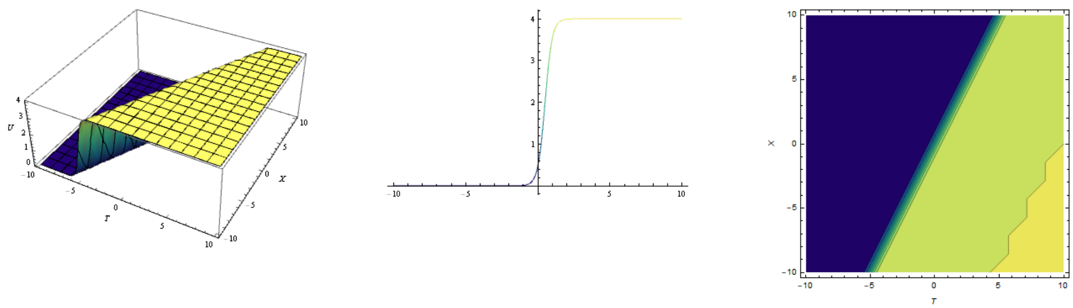

Figure 4.31. 3D, 2D, Contour Plots of the solution (4.5.12) for B=R=1, where $W(T) = 0$.

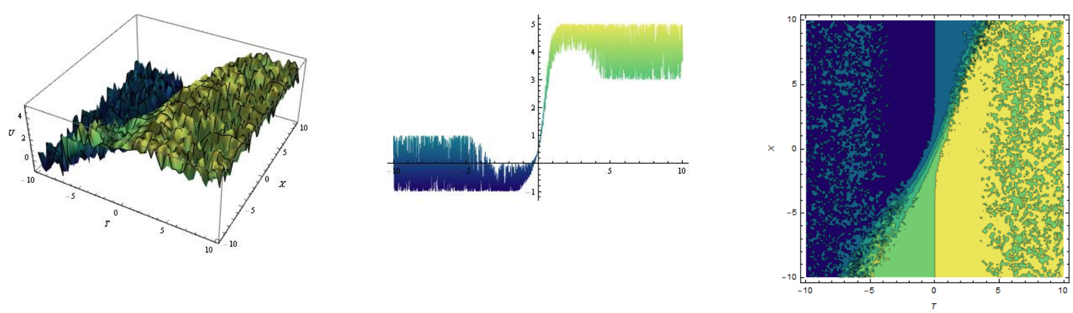

Figure 4.32. 3D, 2D, Contour Plots of the solution (4.5.12) for B=R=1, where $W(T) = \sin[noise * T]$.

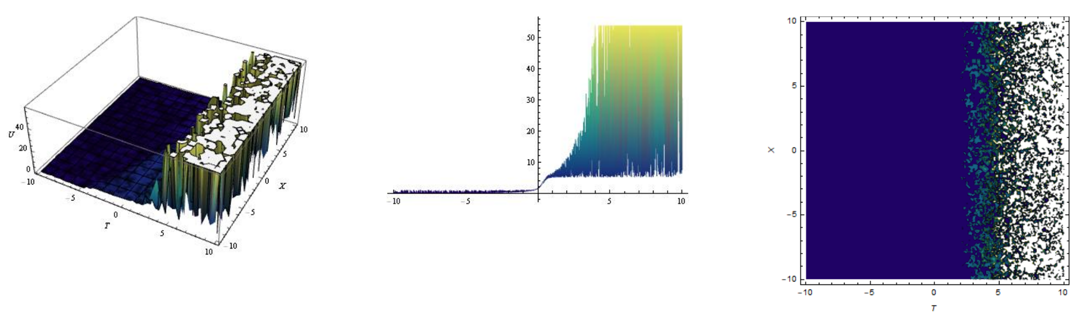

Figure 4.33. 3D, 2D, Contour Plots of the solution (4.5.12) for B=R=1, where $W(T) = exp(noise * T)$.

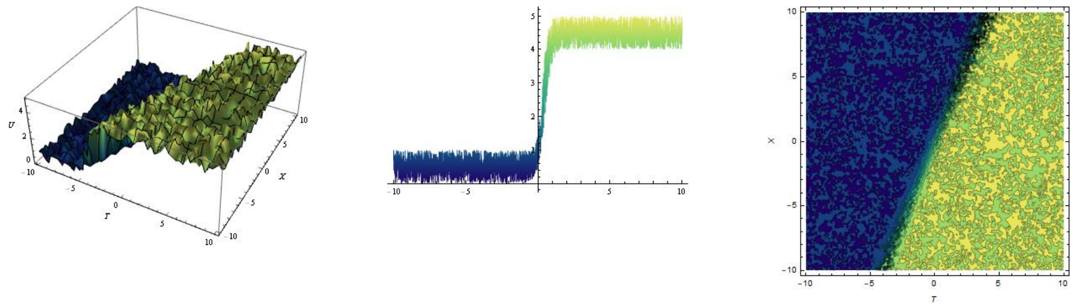

Figure 4.34. 3D, 2D, Contour Plots of the solution (4.5.12) for B=R=1, where $W(T) = noise * 1$.

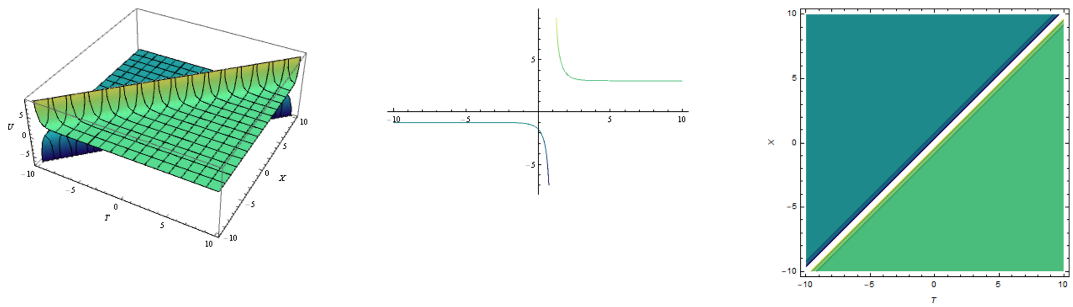

Figure 4.35. 3D, 2D, Contour Plots of the solution (4.5.22) for B=R=1, where $W(T) = 0$.

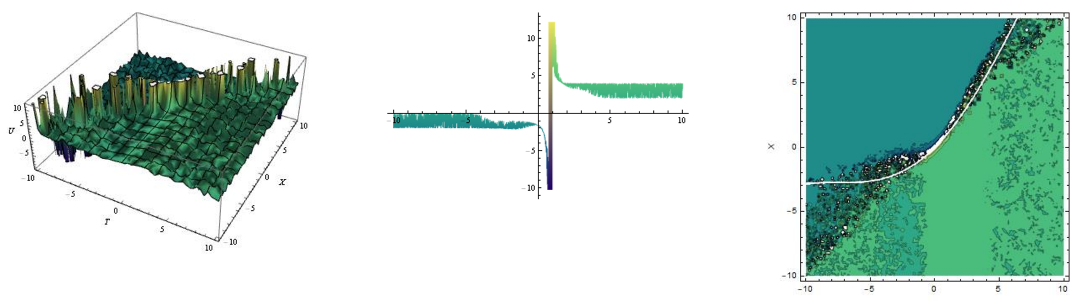

Figure 4.36. 3D, 2D, Contour Plots of the solution (4.5.22) for B=R=1, where $W(T) = \sin[noise * T]$.

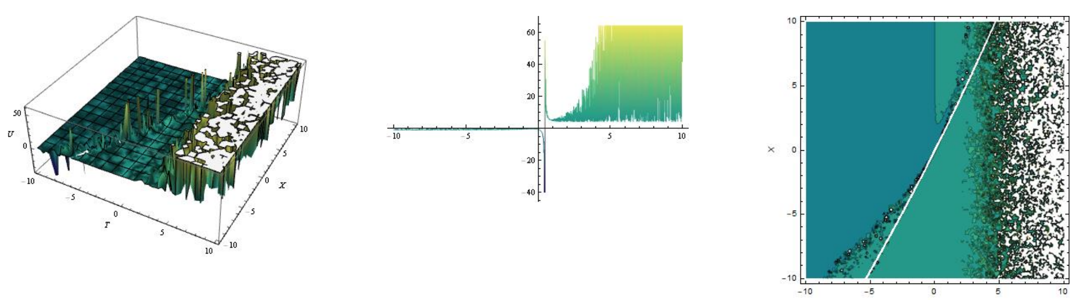

Figure 4.37. 3D, 2D, Contour Plots of the solution (4.5.22) for B=R=1, where $W(T) = exp(noise * T)$.

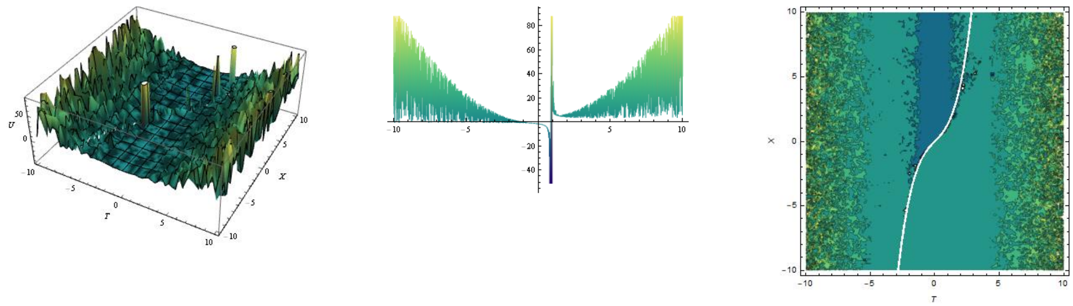

Figure 4.38. 3D, 2D, Contour Plots of the solution (4.5.22) for B=R=1, where $W(T) = noise * T^2$.

4.6. Using Galilean Transform for Solving the Stochastic Burgers' Equation Via the Method of Extended Tanh

4.6.1. The method of extended tanh with zero boundary condition

Based on what was previously found, the balancing parameter takes the value $M = 1$.

Hence using (3.2.9) the next step of the method of extended tanh gives the following finite expansion

$$V(z) = a_0 + a_1 Y + a_{-1} Y^{-1}, Y = tanh(z) \tag{4.6.1}$$

From the two relations (4.6.1) and (4.5.5), we get to the following system of algebraic equations for a_0, a_1, a_{-1}, k and c after collecting the $Y^S, (S = 0,1, \dots ,4)$ coefficients to each other and equating them with zero

$$
\begin{aligned}
&Y^0 .: \quad \tfrac{1}{2} a_{-1}^2 + Bka_{-1} = 0 \\
&Y^1 .: \quad a_{-1}a_0 - ca_{-1} = 0 \\
&Y^2 .: \quad a_{-1}a_1 - ca_0 - Bka_1 + \frac{1}{2} a_0^2 - Bka_{-1} = 0. \\
&Y^3 .: \quad - ca_1 + a_0 a_1 = 0 \\
&Y^4 .: \quad \tfrac{1}{2} a_1^2 + Bka_1
\end{aligned}
\tag{4.6.2}
$$

Now, using one of the symbolic calculation programs to solve the above algebraic system, we get the following two cases of solutions:

1.

$$c = 4Bk, \quad k = k, \quad a_{-1} = -2Bk, \quad a_0 = 4Bk, \quad a_1 = -2Bk. \tag{4.6.3}$$

2.

$$c = -2Bk, \quad k = k, \quad a_{-1} = -2Bk, \quad a_0 = -2Bk, \quad a_1 = 0. \tag{4.6.4}$$

Solutions of the Case 1. Substituting the solutions (4.6.3) into (3.2.9) and using (3.2.6), (3.2.2), we obtain the following soliton solution;

$$\begin{aligned} u_1(x,t) &= 2Bk\coth\,[k(2Bkt - x)](\tanh\,[k(2Bkt - x)] + 1)^2 \\ &= -2Bk\coth\,[k(x - 2Bkt)](1 - \tanh\,[k(x - 2Bkt)])^2 \end{aligned} \tag{4.6.5}$$

From (4.6.5) and Eqs. (4.5.2)- (4.5.3), we arrive at a set of exact stochastic solutions of Eq. (4.5.1), which are simplified as follows:

$$U_1(X,T) = 2Bk\coth[\phi_1(X,T)]\,(\tanh[\phi_1(X,T)] + 1)^2 + W(T), \tag{4.6.6}$$

where

$$\phi_1(X,T) = \left[k\left(2BkT - X + \int_0^T W(T')d\,T'\right)\right]$$

Solutions of the Case 2. Substituting the solutions (4.6.4) into (3.2.9) and using (3.2.6), (3.2.2), we obtain the following soliton solutions;

$$u_2(x,t) = -2Bk\coth[k(2Bkt + x)]\,(1 + \tanh[k(2Bkt + x)]). \tag{4.6.7}$$

From (4.6.7) and Eqs. (4.5.2)- (4.5.3), we arrive at a set of exact stochastic solutions of Eq. (4.5.1), which are in the form of shock wave and simplified as follows:

$$U_2(X,T) = -2Bk\coth[\phi_2(X,T)]\,(1 + \tanh[\phi_2(X,T)]) + W(T), \tag{4.6.8}$$

Where $\phi_2(X,T) = \left[k\left(2BkT + X - \int_0^T W(T')d\,T'\right)\right]$

4.6.2. The method of extended tanh without boundary condition

Based on what was previously found, the balancing parameter takes the value $M = 1$.

Hence using (3.2.9) the next step of the method of extended tanh gives the following finite expansion

$$V(z) = a_0 + a_1 Y + a_{-1} Y^{-1}, Y = tanh(z). \tag{4.6.9}$$

From the two relations (4.6.9) and (4.5.13), we get to the following system of algebraic equations for a_0, a_1, a_{-1}, k and c after collecting the $Y^S, (S = 0,1, \dots ,6)$ coefficients to each other and equating them with zero

$$\begin{aligned}
&Y^0 .: \quad -2Ba_{-1}k - a_{-1}^2 = 0 \\
&Y^1 .: \quad ca_{-1} - a_{-1}a_0 = 0 \\
&Y^2 .: \quad 2Ba_{-1}k + d_{-1}^2 = 0 \\
&Y^3 .: \quad -ca_{-1} - ca_1 + a_{-1}a_0 + a_0 a_1 = 0. \\
&Y^4 .: \quad 2Bka_1 + a_1^2 = 0 \\
&Y^5 .: \quad ca_1 - a_0 a_1 = 0 \\
&Y^6 .: \quad -2Bka_1 - a_1^2 = 0
\end{aligned} \tag{4.6.10}$$

Now, using one of the symbolic calculation programs such as Maple and mathematica to solve the above algebraic system, we get the following solution:

$$c = a_0, \quad k = k, \quad a_{-1} = -2Bk, \quad a_0 = a_0, \quad a_1 = -2Bk. \tag{4.6.11}$$

Substituting the solutions (4.6.11) into (3.2.9) and using (3.2.6), (3.2.2), we arrive to the following solitons solution;

$$u(x,t) = a_0 + 2Bk(\coth[k(a_0 t - x)] + \tanh[k(a_0 t - x)]), \tag{4.6.12}$$

or

$$u(x,t) = a_0 + 4Bk \coth[2k(a_0 t - x)]\,, \tag{4.6.13}$$

where the arbitrary constant a_0 affects the solution (and therefore its boundary condition) as well as. From (4.6.12)-(4.6.13) and Eqs. (4.5.2)- (4.5.3), we arrive at a set of exact stochastic solutions of Eq. (4.5.1), which are simplified as follows:

$$U(X,T) = a_0 + 2Bk(\coth[\phi(X,T)] + \tanh[\phi(X,T)]) + W(T), \tag{4.6.14}$$

or

$$U(X,T) = a_0 + 4Bk \coth[2\phi(X,T)] + W(T), \tag{4.6.15}$$

where

$$\phi(X,T) = \left[k\left(a_0 T - X - \int_0^T W(T')\right)\right]$$

Remark 6. In (4.6.15) if a is chosen appropriately we obtain solution (4.6.6).

4.6.3. Visualization of some solutions

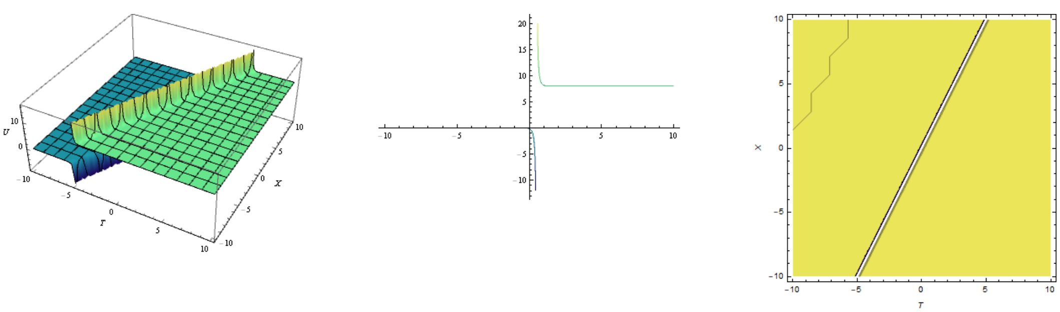

Figure 4.39. 3D, 2D, Contour Plots of the solution (4.6.6) for B=R=1, where $W(T) = 0$.

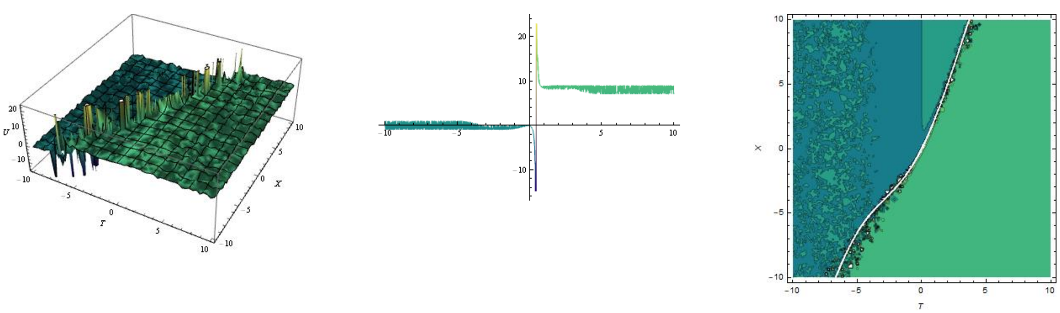

Figure 4.40. 3D, 2D, Contour Plots of the solution (4.6.6) for B=R=1, where $W(T) = \sin[noise * T]$.

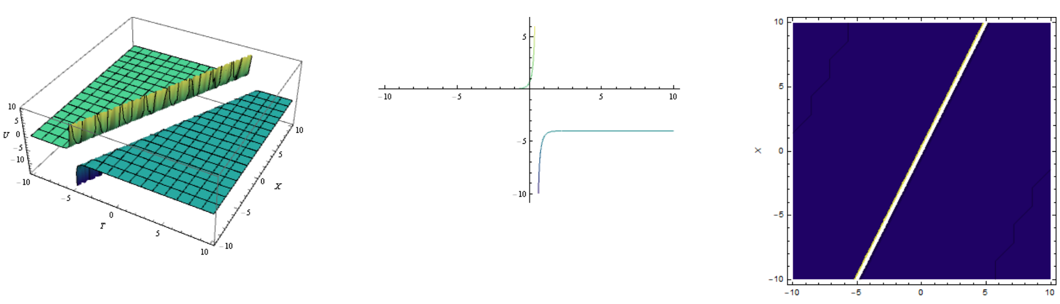

Figure 4.41. 3D, 2D, Contour Plots of the solution (4.6.8) for B=R=1, where $W(T) = 0$.

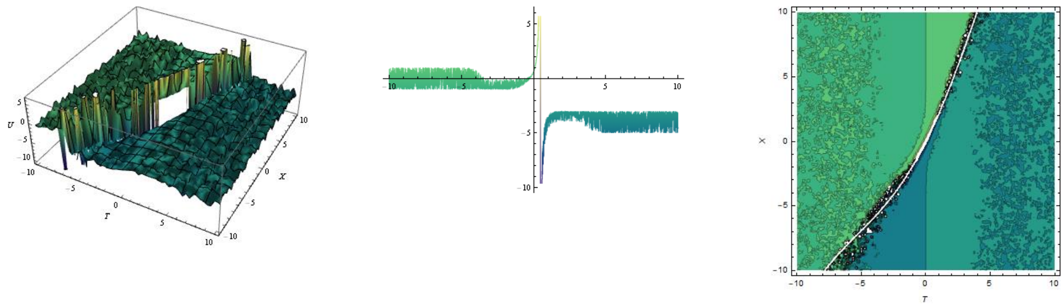

Figure 4.42. 3D, 2D, Contour Plots of the solution (4.6.8) for B=R=1, where $W(T) = \sin[noise * T]$.

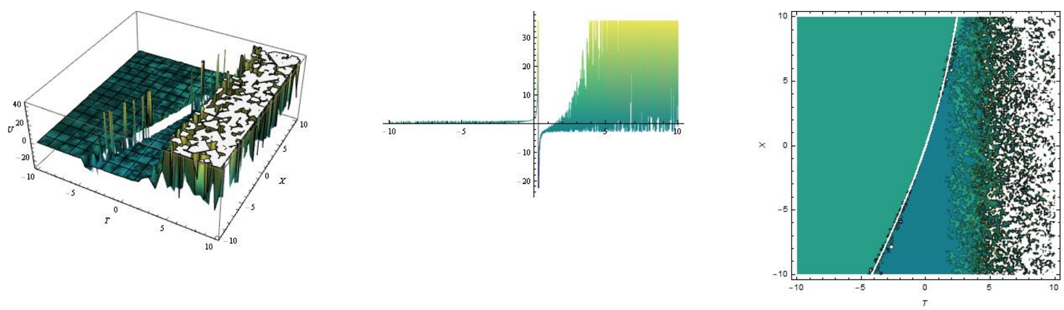

Figure 4.43. 3D, 2D, Contour Plots of the solution (4.6.8) for B=R=1, where $W(T) = exp(noise * T)$.

4.7. Using Galilean Transform for Solving The Stochastic Kuramoto - Sivashinsky (KS) Equation Via The Method of Tanh

The present section is concerned with solitary wave solutions (solutions preserving their shapes as they travel with a phase speed c) of Kuramoto-Sivashinsky (KS)

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} = 0, \tag{4.7.1}$$

Linear terms in the KS equation describe the balance between short wave stability and long wave instability while nonlinear terms provide a mechanism for transferring energy between wave modes (Sajjadian, 2014). The equation that is studied in the present section is as follows:

$$U_t + AUU_X + BU_{XX} + RU_{XXXX} = \eta(T), \tag{4.7.2}$$

where A, B and R are arbitrary constants, $\eta(.)$ stands for the external noise and the subscripts represent the partial derivatives with respect to X and T. Eq. (4.7.2) arises in the modeling of erosion processes by ion sputtering in the surface of amorphous materials (Cuerno et al., 1995). We start by applying the Galilean transformation to Eq. (4.7.2)

$$U(X,T) = u(x,t) + W(T), x = X + m(t), t = T, \tag{4.7.3}$$

$$m(T) = -A\int_0^T W(T')dT', \quad W(T) = \int_0^T \eta(T')dT', \tag{4.7.4}$$

to transform the stochastic Kuramoto—Sivashinsky equation into its deterministic counterpart

$$u_t + Auu_x + Bu_{xx} + Ru_{xxxx} = 0. \tag{4.7.5}$$

In the following, we use the tanh method to obtain solitary wave solutions to the stochastic Kuramoto—Sivashinsky equation (4.7.2). Then by using the change of variable $u(x,t) = V(z), z = k(x - ct)$ and integrating the resulting equation, the KS equation (4.7.5) transforms into an ODE of the form

$$-cV + \frac{A}{2}V^2 + BkV' + Rk^3V''' = 0, \tag{4.7.6}$$

where the constant of integration is set to zero. By balancing $V'''$with V^2 (4.7.6), we have

$$M + 3 = 2M, \tag{4.7.7}$$

hence

$$M = 3. \tag{4.7.8}$$

Consequently, we use the following solution expansion for the tanh method

$$V(z) = a_0 + a_1Y + a_2Y^2 + a_3Y^3, Y = tanh(z). \tag{4.7.9}$$

If we use (4.7.9) in (4.7.6) and set the coefficients of different degree of Y^S, $(S = 0,1,\dots,6)$ to zero, we get the following system of algebraic equations for a_0, a_1, a_2, a_3, k and c:

Y^0 .: $\quad -ca_0 + \frac{1}{2}Aa_0^2 + 6a_3Rk^3 - 2Rk^3a_1 + Bka_1 = 0$

Y^1 .: $\quad -16Rk^3a_2 + Aa_0a_1 + 2Bka_2 - ca_1 = 0$

Y^2 .: $\quad 3Bka_3 + Aa_0a_2 - 60a_3Rk^3 + 8Rk^3a_1 - Bka_1 - ca_2 + \frac{1}{2}Aa_1^2 = 0$

Y^3 .: $\quad 40Rk^3a_2 + Aa_0a_3 + Aa_1a_2 - 2Bka_2 - ca_3 = 0.$ (4.7.10)

Y^4 .: $\quad Aa_1a_3 + 114a_3Rk^3 - 6Rk^3a_1 - 3Bka_3 + \frac{1}{2}Aa_2^2 = 0$

Y^5 .: $\quad -24Rk^3a_2 + Aa_2a_3 = 0$

Y^6 .: $\quad -60a_3Rk^3 + \frac{1}{2}Aa_3^2 = 0$

The following two cases of solutions are found by solving the above algebraic system via symbolic calculation programs:

1.

$$a_0 = \frac{30B}{19A}\sqrt{\frac{-B}{19R}}, a_1 = \frac{45B}{19A}\sqrt{\frac{-B}{19R}}, a_2 = 0, a_3 = -\frac{15B}{19A}\sqrt{\frac{-B}{19R}}, k = \frac{1}{2}\sqrt{\frac{-B}{19R}}, c = \frac{30B}{19}\sqrt{\frac{-B}{19R}}, \quad \frac{B}{R} < 0. \qquad (4.7.11)$$

2.

$$a_0 = \frac{30B}{19A}\sqrt{\frac{11B}{19R}}, a_1 = -\frac{135B}{19A}\sqrt{\frac{11B}{19R}}, a_2 = 0, a_3 = -\frac{165B}{19A}\sqrt{\frac{11B}{19R}}, k = \frac{1}{2}\sqrt{\frac{11B}{19R}}, c = \frac{30B}{19}\sqrt{\frac{11B}{19R}}, \quad \frac{B}{R} > 0. \quad (4.7.12)$$

Solutions of the case 1. Substituting the solutions (4.7.11) into (3.2.8) and using(3.2.6), (3.2.2), we get the following soliton solutions for $\frac{B}{R} < 0$;

$$u_1(x,t) = \frac{15B}{19A}\sqrt{\frac{-B}{19R}}(2 + 3\tanh[\varphi(x,t)] - \tanh^3[\varphi(x,t)]), \qquad (4.7.13)$$

and

$$u_2(x,t) = \frac{15B}{19A}\sqrt{\frac{-B}{19R}}(2 + 3\coth[\varphi(x,t)] - \coth^3[\varphi(x,t)]), \qquad (4.7.14)$$

where

$$\varphi(x,t) = \frac{1}{2}\sqrt{\frac{-B}{19R}}\left(x - \frac{30B}{19}\sqrt{\frac{-B}{19R}}t\right)$$

However, for $\frac{B}{R} > 0$, complex solutions can be easily obtained. From (4.7.13)-(4.7.14) and Eqs. (4.7.3)- (4.7.4), we arrive at a set of exact stochastic solutions of Eq. (4.7.2), which are simplified as follows:

$$U_1(X,T) = \frac{15B}{19A}\sqrt{\frac{-B}{19R}}(2 + 3\tanh[\phi_1(X,T)] - \tanh^3[\phi_1(X,T)]) + W(T). \quad (4.7.15)$$

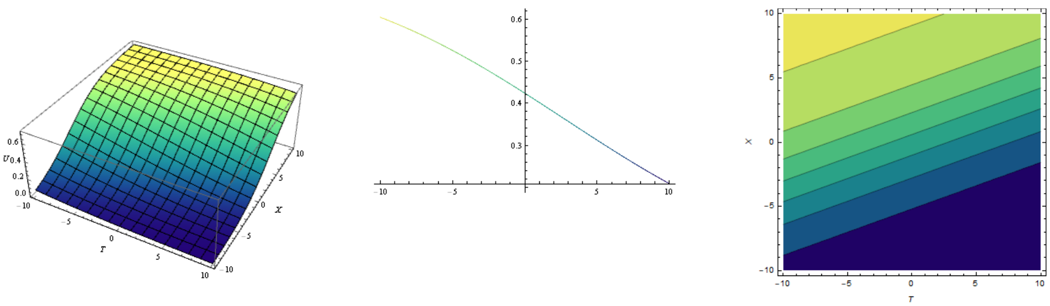

Figure 4.44. 3D, 2D, Contour Plots of the solution (4.7.15) for $B = -1, A = R = 1$, where $W(T) = 0$.

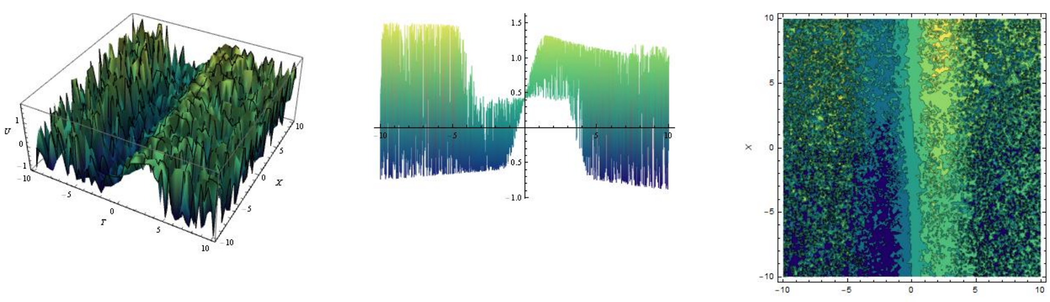

Figure 4.45. 3D, 2D, Contour Plots of the solution (4.7.15) for B=-1,A=R=1, $W(T) = \sin[noise * T]$.

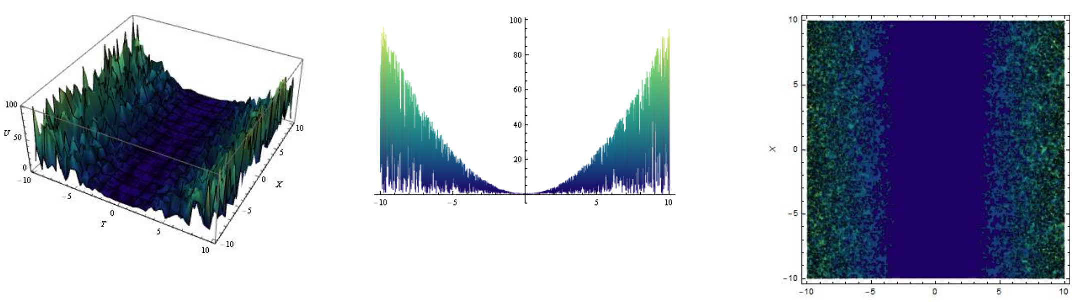

Figure 4.46. 3D, 2D, Contour Plots of the solution (4.7.15) for B=-1,A=R=1, $W(T) = noise * T^2$.

and

$$U_2(X,T) = \frac{15B}{19A}\sqrt{\frac{-B}{19R}}\left(2 + 3\coth[\phi_2(X,T)] - \coth^3[\phi_2(X,T)]\right) + W(T), \quad (4.7.16)$$

where

$$\phi_1(X,T) = \phi_2(X,T) = \left[\frac{1}{2}\sqrt{\frac{-B}{19R}}\left(X - A\int_0^T W(T')dT' - \frac{30B}{19}\sqrt{\frac{-B}{19R}}T\right)\right]$$

Solutions of the Case 2. Inserting the solutions (4.7.12) into (3.2.8) and using(3.2.6),(3.2.2), we have the soliton solutions for $\frac{B}{R} > 0$;

$$u_3(x,t) = \frac{15B}{19A}\sqrt{\frac{11B}{19R}}\left(2 - 9\tanh[\varphi(x,t)] + 11\tanh^3[\varphi(x,t)]\right), \quad (4.7.17)$$

and

$$u_4(x,t) = \frac{15B}{19A}\sqrt{\frac{11B}{19R}}\left(2 - 9\coth[\varphi(x,t)] + 11\coth^3[\varphi(x,t)]\right), \quad (4.7.18)$$

where

$$\varphi(x,t) = \frac{1}{2}\sqrt{\frac{11B}{19R}}\left(x - \frac{30B}{19}\sqrt{\frac{11B}{19R}}t\right).$$

As for the previous case, complex solutions can be easily obtained for $\frac{B}{R} < 0$,.

From (4.7.17)-(4.7.18) and Eqs. (4.7.3)-(4.7.4), we get a series of exact stochastic solutions of Eq. (4.7.2) which are simplified as follows:

$$U_3(X,T) = \frac{15B}{19A}\sqrt{\frac{11B}{19R}}\left(2 - 9\tanh[\phi_3(X,T)] + 11\tanh^3[\phi_3(X,T)]\right) + W(T), \quad (4.7.19)$$

and

$$U_4(X,T)=\frac{15B}{19A}\sqrt{\frac{11B}{19R}}\left(2-9\coth[\phi_4(X,T)]+11\coth^3[\phi_4(X,T)]\right)+W(T), \quad (4.7.20)$$

where

$$\phi_3(X,T)=\phi_4(X,T)=\left[\frac{1}{2}\sqrt{\frac{11B}{19R}}\left(X-A\int_0^T W(T')dT'-\frac{30B}{19}\sqrt{\frac{11B}{19R}}T\right)\right]$$

4.8. Using Galilean Transform for Solving the Stochastic Kuramoto - Sivashinsky (KS) Equation Via the Method of Extended-Tanh

Recalling that $M = 3$ and using (3.2.9), we have the following finite expansion for the extended-tanh method

$$V(z)=a_0+a_1Y+a_2Y^2+a_3Y^3+a_{-1}Y^{-1}+a_{-2}Y^{-2}+a_{-3}Y^{-3}, Y=tanh(z). \quad (4.8.1)$$

Inserting (4.8.1) into(4.7.6), setting the coefficients of the same degree of $Y^S, (S=0,1,\dots,12)$ to zero, we get to the following system of algebraic equations for $a_0, a_1, a_2, a_3, a_{-1}, a_{-2}, a_{-3}, k$ and c:

Y^0 .: $-60Rk^3a_{-3}+\frac{1}{2}Aa_{-3}^2=0$

Y^1 .: $-24Rk^3a_{-2}+Aa_{-3}a_{-2}=0$

Y^2 .: $114Rk^3a_{-3}-6Rk^3a_{-1}+Aa_{-3}a_{-1}+\frac{1}{2}Aa_{-2}^2-3Bka_{-3}=0,$ (4.8.2)

Y^3 .: $40Rk^3a_{-2}+Aa_{-3}a_0+Aa_{-2}a_{-1}-2Bka_{-2}-ca_{-3}=0$

Y^4 .: $-60Rk^3a_{-3}+8Rk^3a_{-1}+Aa_{-3}a_1+Aa_{-2}a_0+\frac{1}{2}Aa_{-1}^2+3Bka_{-3}-Bka_{-1}-ca_{-2}=0,$

Y^5 .: $-16Rk^3a_{-2}+Aa_{-3}a_2+Aa_{-2}a_1+Aa_{-1}a_0+2Bka_{-2}-ca_{-1}=0,$

Y^6 .: $6Rk^3a_{-3}-2Rk^3a_{-1}-2Rk^3a_1+6Rk^3a_3+Aa_{-3}a_3+Aa_{-2}a_2+Aa_{-1}a_1+\frac{1}{2}Aa_0^2+Bka_{-1}+Bka_1-ca_0=0,$

Y^7 .: $-16Rk^3a_2+Aa_{-2}a_3+Aa_{-1}a_2+Aa_0a_1+2Bka_2-ca_1=0,$

Y^8 .: $\quad 8Rk^3a_1 - 60Rk^3a_3 + Aa_{-1}a_3 + Aa_0a_2 + \frac{1}{2}Aa_1^2 - Bka_1 + 3Bka_3 - ca_2 = 0,$

Y^9 .: $\quad 40Rk^3a_2 + Aa_0a_3 + Aa_1a_2 - 2Bka_2 - ca_3 = 0,$

Y^{10} .: $\quad -6Rk^3a_1 + 114Rk^3a_3 + Aa_1a_3 + \frac{1}{2}Aa_2^2 - 3Bka_3 = 0,$

Y^{11} .: $\quad -24Rk^3a_2 + Aa_2a_3 = 0,$

Y^{12} .: $\quad -60Rk^3a_3 + \frac{1}{2}Aa_3^2 = 0$

Two cases of solutions are obtained by solving the above system via symbolic calculation programs.

1.

$$a_0 = \frac{30B}{19A}\sqrt{\frac{-B}{19R}}, \quad a_1 = \frac{135B}{152A}\sqrt{\frac{-B}{19R}}, \quad a_2 = 0, \quad a_3 = -\frac{15B}{152A}\sqrt{\frac{-B}{19R}}, \quad a_{-1} = \frac{135B}{152A}\sqrt{\frac{-B}{19R}},$$
$$a_{-2} = 0, a_{-3} = -\frac{15B}{152A}\sqrt{\frac{-B}{19R}}, \quad k = \frac{1}{4}\sqrt{\frac{-B}{19R}}, \quad c = \frac{30B}{19}\sqrt{\frac{-B}{19R}}, \quad \frac{B}{R} < 0, \tag{4.8.3}$$

2.

$$a_0 = \frac{30B}{19A}\sqrt{\frac{11B}{19R}}, \quad a_1 = -\frac{45B}{152A}\sqrt{\frac{11B}{19R}}, \quad a_2 = 0, \quad a_3 = \frac{165B}{152A}\sqrt{\frac{11B}{19R}},$$
$$a_{-1} = -\frac{45B}{152A}\sqrt{\frac{11B}{19R}}, \quad a_{-2} = 0, a_{-3} = \frac{165B}{152A}\sqrt{\frac{11B}{19R}}, \quad k = \frac{1}{4}\sqrt{\frac{11B}{19R}},$$
$$c = \frac{30B}{19}\sqrt{\frac{11B}{19R}}, \quad \frac{B}{R} > 0, \tag{4.8.4}$$

Solutions of the case 1. Inserting the solutions (4.8.3) into (3.2.9) and using (3.2.6),(3.2.2), we arrive to the following soliton solutions for $\frac{B}{R} < 0$,

$$u(x,t) = \frac{15B}{152A}\sqrt{\frac{-B}{19R}}(16 + 9\tanh[\varphi] - \tanh^3[\varphi] + 9\coth[\varphi] - \coth^3[\varphi]), \tag{4.8.5}$$

where

$$\varphi = k(x - ct), k = \frac{1}{4}\sqrt{\frac{-B}{19R}}, \quad c = \frac{30B}{19}\sqrt{\frac{-B}{19R}}, \quad \frac{B}{R} < 0$$

However, complex solutions are obtained for $\frac{B}{R} > 0$. From (4.8.5) and Eqs. (4.7.3)-(4.7.4), we arrive at a set of exact stochastic solutions of Eq. (4.7.2), which are simplified as follows:

$$U(X,T) = \frac{15B}{152A}\sqrt{\frac{-B}{19R}}(16 + 9\tanh[\phi] - \tanh^3[\phi] + 9\coth[\phi] - \coth^3[\phi]) + W(T), \quad (4.8.6)$$

where

$$\phi = \phi(X,T) = \left[\frac{1}{4}\sqrt{\frac{-B}{19R}}\left(X - A\int_0^T W(T')dT' - \frac{30B}{19}\sqrt{\frac{-B}{19R}}T\right)\right]$$

Solutions of the Case 2. Substituting the solutions (4.8.4) into (3.2.9) and using (3.2.6),(3.2.2), we arrive to the following soliton solution for $\frac{B}{R} > 0$;

$$u(x,t) = \frac{15B}{152A}\sqrt{\frac{11B}{19R}}(16 - 3\tanh[\varphi] + 11\tanh^3[\varphi] - 3\coth[\varphi] + 11\coth^3[\varphi]), (4.8.7)$$

where

$$\varphi = k(x - ct), k = \frac{1}{4}\sqrt{\frac{11B}{19R}}, \quad c = \frac{30B}{19}\sqrt{\frac{11B}{19R}}, \quad \frac{B}{R} > 0$$

However, for $\frac{B}{R} < 0$, complex solutions can be easily obtained. From (4.8.7)and Eqs. (4.7.3)-(4.7.4), we arrive at a set of exact stochastic solutions of Eq. (4.7.2), which are simplified as follows:

$$U(X,T) = \frac{15B}{152A}\sqrt{\frac{11B}{19R}}(16 - 3\tanh[\phi] + 11\tanh^3[\phi] - 3\coth[\phi] + 11\coth^3[\phi]) + W(T), \quad (4.8.8)$$

where $\phi = \phi(X,T) = \left[\frac{1}{4}\sqrt{\frac{11B}{19R}}\left(X - A\int_0^T W(T')dT' - \frac{30B}{19}\sqrt{\frac{11B}{19R}}T\right)\right]$

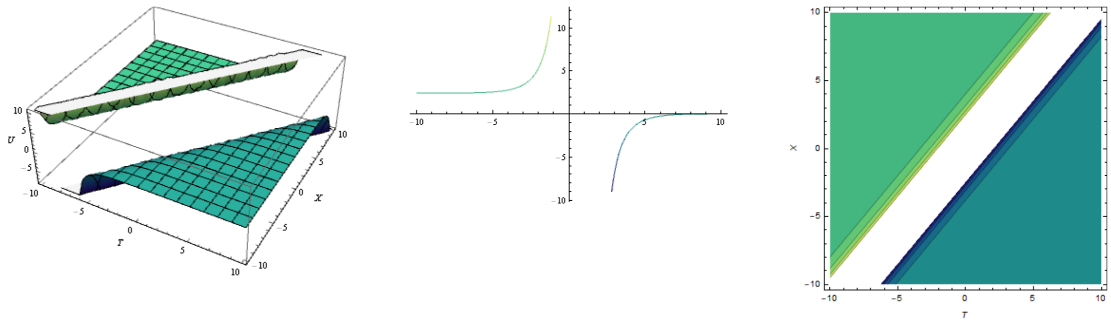

Figure 4.47. 3D, 2D, Contour Plots of the solution (4.8.8) for B=A=R=1, where $W(T) = 0$.

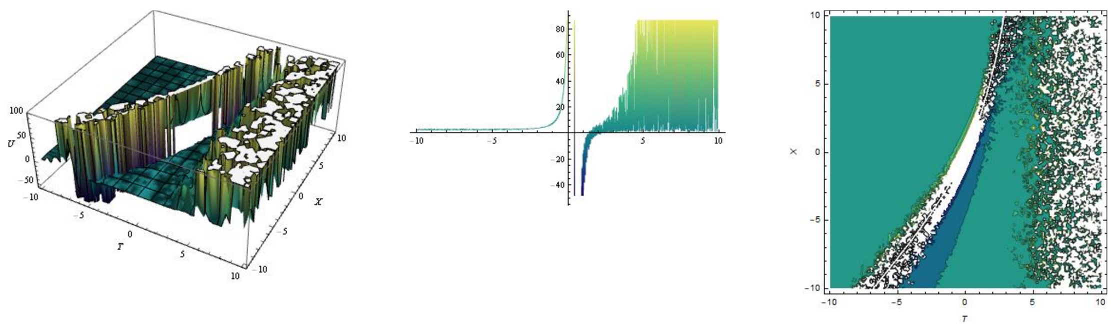

Figure 4.48. 3D, 2D, Contour Plots of the solution (4.8.8) for B=A=R=1, where $W(T) = exp(noise * T)$.

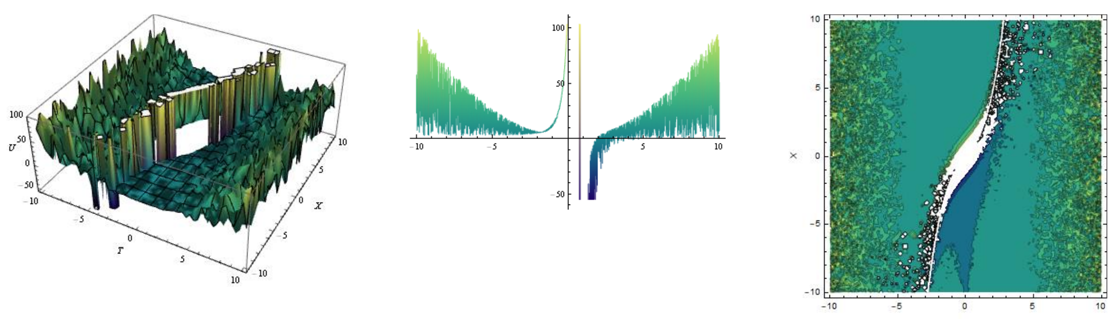

Figure 4.49. 3D, 2D, Contour Plots of the solution (4.8.8) for B=A=R=1, where $W(T) = noise * T^2$.

4.9. Using Galilean Transform for Solving the Stochastic Kawahara (KH) Equation Via the Method of Tanh

The present section is concerned with solitary wave solutions (solutions preserving their shapes as they travel with a phase speed c) of Kawahara equation

$$u_t + uu_x + u_{xxx} + \nu u_{xxxxx} = 0, \tag{4.9.1}$$

with external noise, where ν is positive and represents the viscosity of the system for (4.9.1). Kawahara equation is a dispersive fifth-order equation arising in the modeling of magneto-acoustic waves in plasma and small-amplitude water waves with surface tension and was introduced by Kawahara in 1972 (Kawahara, 1972).

The equation that is studied in the present section is as follows:

$$U_T + AUU_X + BU_{XXX} + RU_{XXXXX} = \eta(T), \qquad (4.9.2)$$

where A, B and R are arbitrary constants, $\eta(.)$ stands for the external noise and the subscripts represent the partial derivatives with respect to X and T. Eq. (4.9.2) arises in the modeling of small-amplitude water waves with surface tension when the surface of the fluid is subjected to a non-constant pressure or when the bottom of the layer is not flat. Let's start from the following equation.

$$U_T + AUU_X + BU_{XXX} + RU_{XXXXX} = \eta(T). \qquad (4.9.3)$$

Here inhomogeneous term $\eta(T)$ stands for external noise and subscripts X and T denote partial differentiations with respect to X and T , respectively. The following Galilean transformation

$$U(X,T) = u(x,t) + W(T), x = X + m(t), t = T, \qquad (4.9.4)$$

$$m(T) = -A\int_0^T W(T')dT', \quad W(T) = \int_0^T \eta(T')dT', \qquad (4.9.5)$$

is applied to transform the stochastic Kawahara equation into its deterministic counterpart

$$u_t + Auu_x + Bu_{xxx} + Ru_{xxxxx} = 0, \qquad (4.9.6)$$

In the following , we use the tanh method to obtain solitary wave solutions to the stochastic Kawahara equation (4.9.2), and then use the extended-tanh method to develop new solitary wave solutions. Then by using the change of variable $u(x,t) = V(z), z = k(x - ct)$ and integrating the resulting equation, the Kawahara equation (4.9.6) transforms into an ODE of the form

$$-cV + \frac{A}{2}V^2 + Bk^2V'' + Rk^4V'''' = 0, \qquad (4.9.7)$$

Balancing V'''' with V^2 in (4.9.7) in (43) yields

$$M + 4 = 2M, \qquad (4.9.8)$$

where we obtain

$$M = 4. \tag{4.9.9}$$

Then, we use the following solution expansion for the tanh method

$$V(z) = a_0 + a_1 Y + a_2 Y^2 + a_3 Y^3 + a_4 Y^4, Y = tanh(z). \tag{4.9.10}$$

From the two relations (4.9.10) and (4.9.7), we get to the following system of algebraic equations for $a_0, a_1, a_2, a_3, a_4, k$ and c after collecting the coefficients of $Y^S, (S = 0,1, \ldots ,8)$ with each other and equating them to zero

Y^0 .: $2Bk^2 a_2 + 24a_4 Rk^4 - 16Rk^4 a_2 - ca_0 + \frac{1}{2} Aa_0^2 = 0$

Y^1 .: $16Rk^4 a_1 - 120Rk^4 a_3 - 2Bk^2 a_1 + 6Bk^2 a_3 + Aa_0 a_1 - ca_1 = 0$

Y^2 .: $-ca_2 + \frac{1}{2} Aa_1^2 + Aa_0 a_2 - 8Bk^2 a_2 + 12Bk^2 a_4 + 136Rk^4 a_2 - 480a_4 Rk^4 = 0$

Y^3 .: $-40Rk^4 a_1 + 576Rk^4 a_3 + 2Bk^2 a_1 - 18Bk^2 a_3 + Aa_0 a_3 + Aa_1 a_2 - ca_3 = 0$

Y^4 .: $\frac{1}{2} Aa_2^2 - ca_4 + Aa_0 a_4 + Aa_1 a_3 - 240Rk^4 a_2 + 6Bk^2 a_2 - 32Bk^2 a_4 + 1696a_4 Rk^4 = 0$

Y^5 .: $24Rk^4 a_1 - 816Rk^4 a_3 + 12Bk^2 a_3 + Aa_1 a_4 + Aa_2 a_3 = 0,$ (4.9.11)

Y^6 .: $-2080a_4 Rk^4 + \frac{1}{2} Aa_3^2 + Aa_2 a_4 + 120Rk^4 a_2 + 20Bk^2 a_4 = 0$

Y^7 .: $360Rk^4 a_3 + Aa_3 a_4 = 0$

Y^8 .: $840a_4 Rk^4 + \frac{1}{2} Ad_4^2 = 0$

Now, using one of the symbolic calculation programs to solve the above algebraic system, we get the following two cases of solutions:

1.

$$a_0 = -\frac{33B^2}{169AR}, \quad a_1 = 0, \quad a_2 = \frac{210B^2}{169AR}, \quad a_3 = 0, \quad a_4 = -\frac{105B^2}{169AR} \quad k = \frac{1}{2}\sqrt{-\frac{B}{13R}}, \quad c = \frac{36B^2}{169R}, \quad \frac{B}{R} < 0. \tag{4.9.12}$$

2.

$$a_0 = -\frac{105B^2}{169AR}, \quad a_1 = 0, \quad a_2 = \frac{210B^2}{169AR}, \quad a_3 = 0, \quad a_4 = -\frac{105B^2}{169AR} \quad k = \frac{1}{2}\sqrt{-\frac{B}{13R}}, \quad c = -\frac{36B^2}{169R}, \quad \frac{B}{R} > 0. \, (4.9.13)$$

Solutions of the Case 1. Substituting the solutions (4.9.12) into (3.2.8) and using (3.2.6),(3.2.2), we arrive to the following solitons solution for $\frac{B}{R} < 0$;

$$u_1(x,t) = -\frac{3B^2}{169AR}(11 - 70\tanh^2[\varphi_1(x,t)] + 35\tanh^4[\varphi_1(x,t)]), \qquad (4.9.14)$$

and

$$u_2(x,t) = -\frac{3B^2}{169AR}(11 - 70\coth^2[\varphi_2(x,t)] + 35\coth^4[\varphi_2(x,t)]), \qquad (4.9.15)$$

where

$$\varphi_{1,2}(x,t) = \frac{1}{2}\sqrt{-\frac{B}{13R}}\left(x - \frac{36B^2}{169R}t\right)$$

Though, we arrive to the following periodic solutions for $\frac{B}{R} > 0$.

$$u_3(x,t) = -\frac{3B^2}{169AR}(11 + 70\tan^2[\varphi_3(x,t)] + 35\tan^4[\varphi_3(x,t)]), \qquad (4.9.16)$$

and

$$u_4(x,t) = -\frac{3B^2}{169AR}(11 + 70\cot^2[\varphi_4(x,t)] + 35\cot^4[\varphi_4(x,t)]), \qquad (4.9.17)$$

where

$$\varphi_{3,4}(x,t) = \frac{1}{2}\sqrt{\frac{B}{13R}}\left(x - \frac{36B^2}{169R}t\right)$$

From (4.9.14) -(4.9.17) and Eqs. (4.9.4) - (4.9.5), we arrive at a set of exact stochastic solutions of Eq. (4.9.2), which are simplified as follows:

$$U_1(X,T) = -\frac{3B^2}{169AR}(11 - 70\tanh^2[\phi_1(X,T)] + 35\tanh^4[\phi_1(X,T)]) + W(T), \quad (4.9.18)$$

and

$$U_2(X,T) = -\frac{3B^2}{169AR}(11 - 70\coth^2[\phi_2(X,T)] + 35\coth^4[\phi_2(X,T)]) + W(T), \quad (4.9.19)$$

where

$$\phi_1(X,T) = \phi_2(X,T) = \left[\frac{1}{2}\sqrt{-\frac{B}{13R}}\left(X - A\int_0^T W(T')dT' - \frac{36B^2}{169R}T\right)\right]$$

For $\frac{B}{R} > 0$:

$$U_3(X,T) = -\frac{3B^2}{169AR}(11 + 70\tan^2[\phi_3(X,T)] + 35\tan^4[\phi_3(X,T)]) + W(T), \quad (4.9.20)$$

and

$$U_4(X,T) = -\frac{3B^2}{169AR}(11 + 70\cot^2[\phi_4(X,T)] + 35\cot^4[\phi_4(X,T)]) + W(T), \quad (4.9.21)$$

where

$$\phi_3(X,T) = \phi_4(X,T) = \left[\frac{1}{2}\sqrt{\frac{B}{13R}}\left(X - A\int_0^T W(T')dT' - \frac{36B^2}{169R}T\right)\right]$$

Solutions of the Case 2. Substituting the solutions (4.9.13) into (3.2.8) and using (3.2.6), (3.2.2), we arrive to the following solitons solution for $\frac{B}{R} < 0$;

$$u_5(x,t) = -\frac{105B^2}{169AR}sech^4\left[\frac{1}{2}\sqrt{-\frac{B}{13R}}\left(x + \frac{36B^2}{169R}t\right)\right], \quad (4.9.22)$$

and

$$u_6(x,t) = -\frac{105B^2}{169AR}cech^4\left[\frac{1}{2}\sqrt{-\frac{B}{13R}}\left(x + \frac{36B^2}{169R}t\right)\right], \quad (4.9.23)$$

The following periodic solutions are obtained for $\frac{B}{R} > 0$.

$$u_7(x,t) = -\frac{105B^2}{169AR}sec^4\left[\frac{1}{2}\sqrt{\frac{B}{13R}}\left(x+\frac{36B^2}{169R}t\right)\right], \tag{4.9.24}$$

and

$$u_8(x,t) = -\frac{105B^2}{169AR}csec^4\left[\frac{1}{2}\sqrt{\frac{B}{13R}}\left(x+\frac{36B^2}{169R}t\right)\right], \tag{4.9.25}$$

From (4.9.22) - (4.9.25) and Eqs. (4.9.4)-(4.9.5), we get a series of exact stochastic solutions of Eq. (4.9.2), which are simplified as follows:

$$U_5(X,T) = -\frac{105B^2}{169AR}sech^4[\phi_5(X,T)] + W(T), \tag{4.9.26}$$

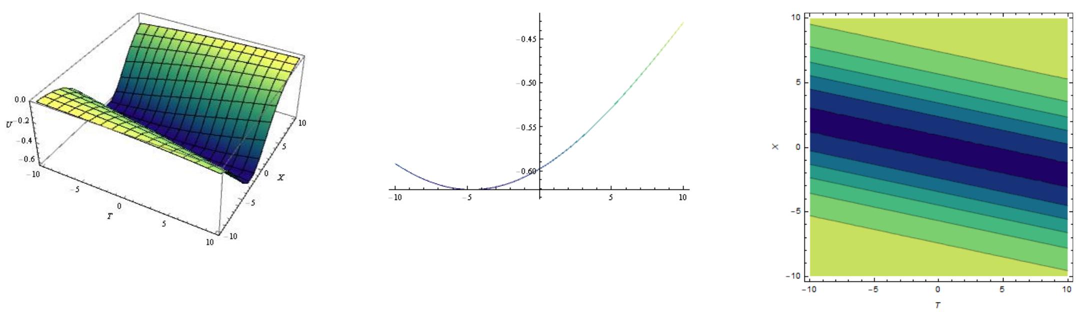

Figure 4.50. 3D, 2D, Contour Plots of the solution (4.9.26) for B=-1,A=R=1,where $W(T) = 0$.

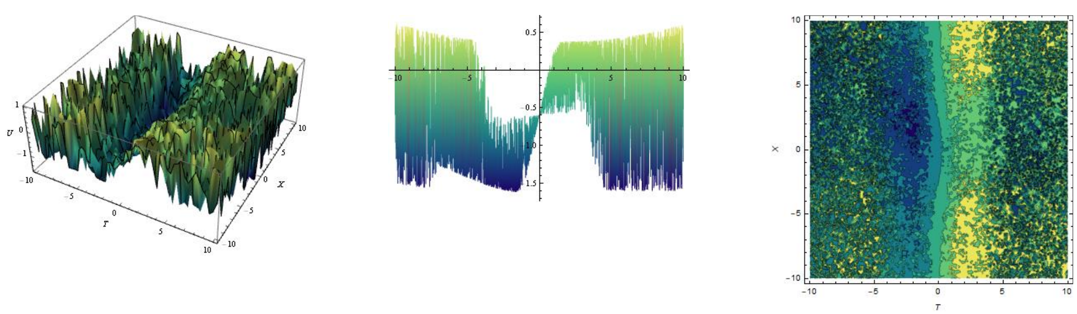

Figure 4.51. 3D, 2D, Contour Plots of the solution (4.9.26) for B=-1,A=R=1, where $W(T) = \sin[\text{noise} * T]$.

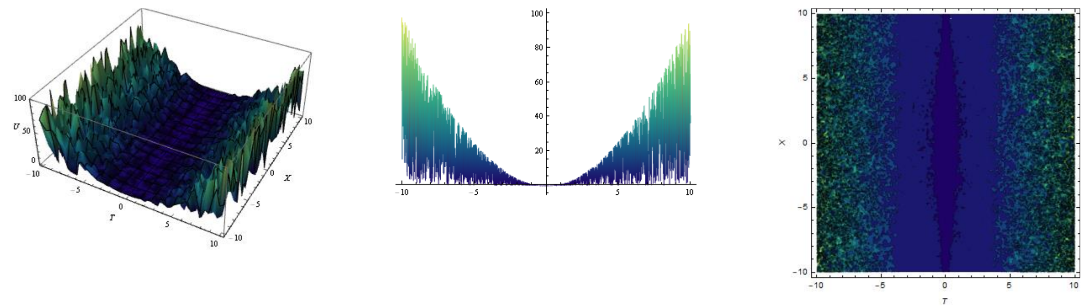

Figure 4.52. 3D, 2D, Contour Plots of the solution (4.9.26) for B=-1,A=R=1,where $W(T)$ = noise $*$ T^2.

and

$$U_6(X,T) = -\frac{105B^2}{169AR} csech^4[\phi_6(X,T)] + W(T), \tag{4.9.27}$$

where

$$\phi_5(X,T) = \phi_6(X,T) = \left[\frac{1}{2}\sqrt{-\frac{B}{13R}}\left(X - A\int_0^T W(T')dT' + \frac{36B^2}{169R}T\right)\right]$$

For $\frac{B}{R} > 0$:

$$U_7(X,T) = -\frac{105B^2}{169AR} sec^4[\phi_7(X,T)] + W(T), \tag{4.9.28}$$

and

$$U_8(X,T) = -\frac{105B^2}{169AR} csec^4[\phi_8(X,T)] + W(T), \tag{4.9.29}$$

where

$$\phi_7(X,T) = \phi_8(X,T) = \left[\frac{1}{2}\sqrt{\frac{B}{13R}}\left(X - A\int_0^T W(T')dT' + \frac{36B^2}{169R}T\right)\right]$$

4.10. Using Galilean Transform for Solving The Stochastic Kawahara (KH) Equation Via The Method of Extended-Tanh

Recalling that $M = 4$ and using (3.2.9), we obtain the following finite expansion for the extended-tanh method

$V(z) = a_0 + a_1 Y + a_2 Y^2 + a_3 Y^3 + a_4 Y^4 + a_{-1} Y^{-1} a_{-2} Y^{-2} + a_{-3} Y^{-3} + a_{-4} Y^{-4}, Y = tanh(z)$, (4.10.1)

Inserting (4.10.1) into (4.9.7) and equating the coefficients of the same degree of $Y^S, (S = 0,1, \ldots ,16)$ yields the system of algebraic equations for a_0, a_1, a_2, a_3, $a_4, a_{-1}, a_{-2}, a_{-3}, a_{-4}, k$ and c:

Y^0 .: $1680Rk^4 a_{-4} + Aa_{-4}^2 = 0,$

Y^1 .: $720Rk^4 a_{-3} + 2Aa_{-4} a_{-3} = 0,$ (4.10.2)

Y^2 .: $-4160Rk^4 a_{-4} + 240Rk^4 a_{-2} + 40Bk^2 a_{-4} + 2Aa_{-4} a_{-2} + Aa_{-3}^2 = 0,$

Y^3 .: $-1632Rk^4 a_{-3} + 48Rk^4 a_{-1} + 24Bk^2 a_{-3} + 2Aa_{-4} a_{-1} + 2Aa_{-3} a_{-2} = 0,$

Y^4 .: $3392Rk^4 a_{-4} - 480Rk^4 a_{-2} - 64Bk^2 a_{-4} + 12Bk^2 a_{-2} + 2Aa_{-4} a_0 + 2Aa_{-3} a_{-1} + Aa_{-2}^2 - 2ca_{-4} = 0,$

Y^5 .: $1152Rk^4 a_{-3} - 80Rk^4 a_{-1} - 36Bk^2 a_{-3} + 4Bk^2 a_{-1} + 2Aa_{-4} a_1 + 2Aa_{-3} a_0 + 2Aa_{-2} a_{-1} - 2ca_{-3} = 0,$

Y^6 .: $-960Rk^4 a_{-4} + 272Rk^4 a_{-2} + 24Bk^2 a_{-4} - 16Bk^2 a_{-2} + 2Aa_{-4} a_2 + 2Aa_{-3} a_1 + 2Aa_{-2} a_0 + Aa_{-1}^2 - 2ca_{-2} = 0,$

Y^7 .: $-240Rk^4 a_{-3} + 32Rk^4 a_{-1} + 12Bk^2 a_{-3} - 4Bk^2 a_{-1} + 2Aa_{-4} a_3 + 2Aa_{-3} a_2 + 2Aa_{-2} a_1 + 2Aa_{-1} a_0 - 2ca_{-1} = 0,$

Y^8 .: $48Rk^4 a_{-4} - 32Rk^4 a_{-2} - 32Rk^4 a_2 + 48Rk^4 a_4 + 4Bk^2 a_{-2} + 4Bk^2 a_2 + 2Aa_{-4} a_4 + 2Aa_{-3} a_3 + 2Aa_{-2} a_2 + 2Aa_{-1} a_1 + Aa_0^2 - 2ca_0 = 0,$

Y^9 .: $32Rk^4 a_1 - 240Rk^4 a_3 - 4Bk^2 a_1 + 12Bk^2 a_3 + 2Aa_{-3} a_4 + 2Aa_{-2} a_3 + 2Aa_{-1} a_2 + 2Aa_0 a_1 - 2ca_1 = 0,$

Y^{10} .: $272Rk^4 a_2 - 960Rk^4 a_4 - 16Bk^2 a_2 + 24Bk^2 a_4 + 2Aa_{-2} a_4 + 2Aa_{-1} a_3 + 2Aa_0 a_2 + Aa_1^2 - 2ca_2 = 0,$

Y^{11} .: $-80Rk^4 a_1 + 1152Rk^4 a_3 + 4Bk^2 a_1 - 36Bk^2 a_3 + 2Aa_{-1} a_4 + 2Aa_0 a_3 + 2Aa_1 a_2 - 2ca_3 = 0,$

Y^{12} .: $-480Rk^4 a_2 + 3392Rk^4 a_4 + 12Bk^2 a_2 - 64Bk^2 a_4 + 2Aa_0 a_4 + 2Aa_1 a_3 + Aa_2^2 - 2ca_4 = 0,$

Y^{13} .: $48Rk^4 a_1 - 1632Rk^4 a_3 + 24Bk^2 a_3 + 2Aa_1 a_4 + 2Aa_2 a_3 = 0,$

Y^{14} .: $240Rk^4 a_2 - 4160Rk^4 a_4 + 40Bk^2 a_4 + 2Aa_2 a_4 + Aa_3^2 = 0,$

Y^{15} .: $720Rk^4 a_3 + 2Aa_3 a_4 = 0,$

Y^{16} .: $1680Rk^4 a_4 + Aa_4^2 = 0,$

Now, using one of the symbolic calculation programs to solve the above algebraic system, we get the following two cases of solutions:

1.

$$a_0 = -\frac{315B^2}{1352AR}, \quad a_1 = 0, \quad a_2 = \frac{105B^2}{676AR}, \quad a_3 = 0, \quad a_4 = -\frac{105B^2}{2704AR} \quad a_{-1} = 0,$$
$$a_{-2} = \frac{105B^2}{676AR}, a_{-3} = 0, \quad a_{-4} = -\frac{105B^2}{2704AR}, \quad k = \frac{1}{4}\sqrt{\frac{-B}{13R}}, \quad c = -\frac{36B^2}{169R}, \quad \frac{B}{R} < 0, \tag{4.10.3}$$

2.

$$a_0 = \frac{261B^2}{1352AR}, \quad a_1 = 0, \quad a_2 = \frac{105B^2}{676AR}, \quad a_3 = 0, \quad a_4 = -\frac{105B^2}{2704AR} \quad a_{-1} = 0,$$
$$a_{-2} = \frac{105B^2}{676AR}, a_{-3} = 0, \quad a_{-4} = -\frac{105B^2}{2704AR} \quad k = \frac{1}{4}\sqrt{\frac{-B}{13R}}, \quad c = \frac{36B^2}{169R}, \quad \frac{B}{R} < 0, \tag{4.10.4}$$

Solutions of the Case 1. We obtain the following soliton solutions for $\frac{B}{R} < 0$ by inserting the solutions (4.10.3) into (3.2.9) and using (3.2.6) , (3.2.2).

$$u_1(x,t) = -\frac{105B^2}{2704AR}(6 - 4\tanh^2[\varphi] + \tanh^4[\varphi] - 4\coth^2[\varphi] + \coth^4[\varphi]), \tag{4.10.5}$$

where

$$\varphi(x,t) = k(x - ct), k = \frac{1}{4}\sqrt{\frac{-B}{13R}}, \quad c = -\frac{36B^2}{169R}, \quad \frac{B}{R} < 0$$

The following periodic solutions are obtained for $\frac{B}{R} > 0$.

$$u_2(x,t) = -\frac{105B^2}{2704AR}(6 + 4\tan^2[\varphi] + \tan^4[\varphi] + 4\cot^2[\varphi] + \cot^4[\varphi]), \tag{4.10.6}$$

where

$$\varphi(x,t) = k(x - ct), k = \frac{1}{4}\sqrt{\frac{B}{13R}}, \quad c = -\frac{36B^2}{169R},$$

From (4.10.5) and Eqs. (4.9.4)-(4.9.5), we have a series of exact stochastic solutions of Eq. (4.9.2), which are simplified as follows:

$$U_1(X,T) = -\frac{105B^2}{2704AR}\left(6 - 4\tanh^2[\phi_1] + \tanh^4[\phi_1] - 4\coth^2[\phi_1] + \coth^4[\phi_1]\right) + W(T), \quad (4.10.7)$$

where

$$\phi_1 = \phi_1(X,T) = \left[\frac{1}{4}\sqrt{\frac{-B}{13R}}\left(X - A\int_0^T W(T')\,dT' + \frac{36B^2}{169R}T\right)\right]$$

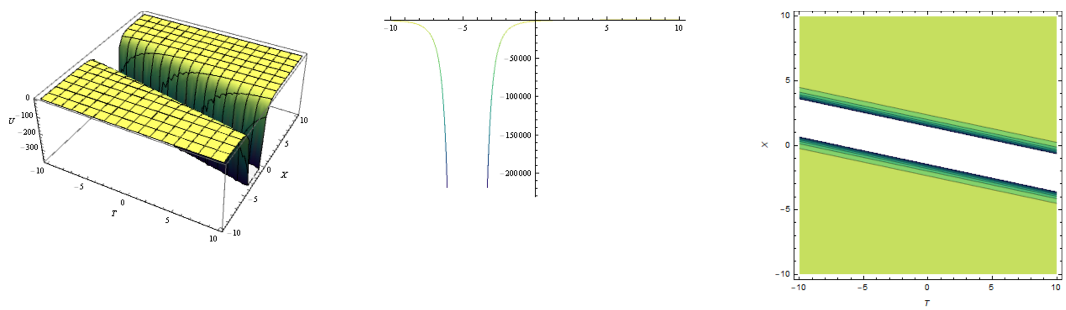

Figure 4.53. 3D, 2D, Contour Plots of the solution (4.10.7) for B=-1,A=R=1, where $W(T)=0$

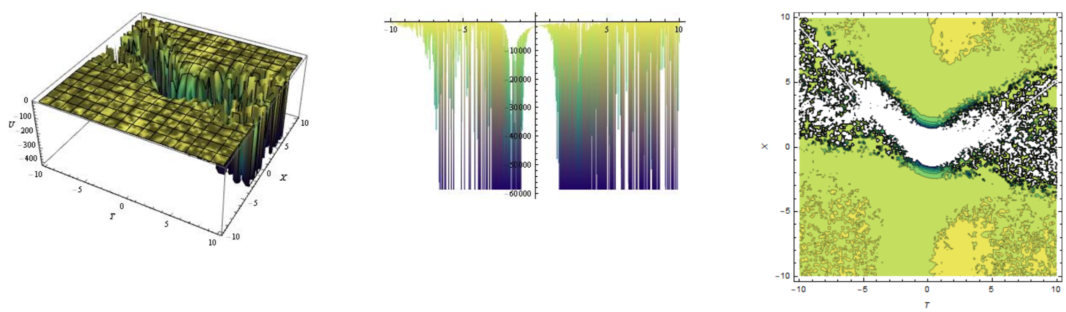

Figure 4.54. 3D, 2D, Contour Plots of the solution (4.10.7) for B=-1,A=R=1, $W(T) = sin[noise * T]$.

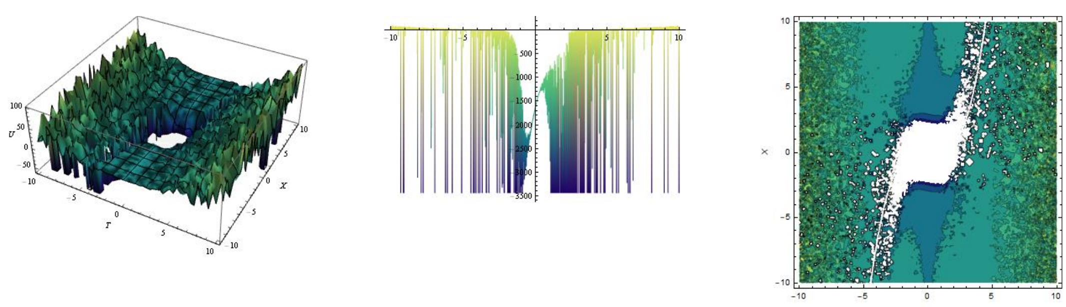

Figure 4.55. 3D, 2D, Contour Plots of the solution (4.10.7) for B=-1,A=R=1, where $W(T) = noise * T^2$.

For $\frac{B}{R} > 0$,

$$U_2(X,T) = -\frac{105B^2}{2704AR}(6 + 4\tan^2[\phi_2] + \tan^4[\phi_2] + 4\cot^2[\phi_2] + \cot^4[\phi_2]) + W(T), \quad (4.10.8)$$

where

$$\phi_2 = \phi_2(X,T) = \left[\frac{1}{4}\sqrt{\frac{B}{13R}}\left(X - A\int_0^T W(T')dT' + \frac{36B^2}{169R}T\right)\right]$$

Solutions of the Case 2. Inserting the solutions (4.10.4) into (3.2.9) and using (3.2.6),(3.2.2), we obtain the soliton solutions for $\frac{B}{R} < 0$

$$u_3(x,t) = \frac{3B^2}{2704AR}(174 + 140\tanh^2[\varphi] - 35\tanh^4[\varphi] + 140\coth^2[\varphi] - 35\coth^4[\varphi]), (4.10.9)$$

where

$$\varphi(x,t) = k(x - ct), k = \frac{1}{4}\sqrt{\frac{-B}{13R}}, \quad c = \frac{36B^2}{169R}, \quad \frac{B}{R} < 0$$

The following periodic solutions are also obtained for $\frac{B}{R} > 0$.

$$u_4(x,t) = \frac{3B^2}{2704AR}(174 - 140\tan^2[\varphi] - 35\tan^4[\varphi] - 140\cot^2[\varphi] - 35\cot^4[\varphi]), (4.10.10)$$

where

$$\varphi(x,t) = k(x - ct), k = \frac{1}{4}\sqrt{\frac{B}{13R}}, \quad c = \frac{36B^2}{169R},$$

From (4.10.9) and Eqs. (4.9.4)-(4.9.5), we arrive at a set of exact stochastic solutions of Eq. (4.9.2), which are simplified as follows:

$$U_3(X,T) = \frac{3B^2}{2704AR}(174 + 140\tanh^2[\phi_3] - 35\tanh^4[\phi_3] + 140\coth^2[\phi_3] - 35\coth^4[\phi_3]) + W(T), \quad (4.10.11)$$

where

$$\phi_3 = \phi_3(X,T) = \left[\frac{1}{4}\sqrt{\frac{-B}{13R}}\left(X - A\int_0^T W(T')dT' - \frac{36B^2}{169R}T\right)\right]$$

For $\frac{B}{R} > 0$,

$$U_4(X,T) = \frac{3B^2}{2704AR}(174 - 140\tan^2[\phi_4] - 35\tan^4[\phi_4] - 140\cot^2[\phi_4] - 35\cot^4[\phi_4]) + W(T), \quad (4.10.12)$$

where

$$\phi_4 = \phi_4(X,T) = \left[\frac{1}{4}\sqrt{\frac{B}{13R}}\left(X - A\int_0^T W(T')dT' - \frac{36B^2}{169R}T\right)\right]$$

4.11. Exact Solutions for Wick-Type Stochastic Extended KdV Equation

In this chapter, we aim to provide exact solutions to the Wick-type extended KdV eqaution

$$U_t + H_1(t) \diamond U_x + H_2(t) \diamond U \diamond U_x + H_3(t) \diamond U_{xxx} = 0, \quad (4.11.1)$$

where $\diamond$ is the Wick product on the Hida distribution space $(S)^*$, and $H_i(i = 1,2,3)$ are the white noise functions. The F-expansion method and Hermit transformation are employed. By means of these methods and with the help of a symbolic computation package, we get periodic wave solutions for the Wick-type stochastic extended KdV equation. 2D, 3D, and contour graphs have been drawn by giving special values to the constants in the solutions via computer software. Moreover, by considering different random values to the noise, the effect of the noise on the wave-forms has been exhibited. The obtained results has been discussed in detail.

The KdV equation was introduced as a model for waves on shallow water surfaces. It is one of the simplest equations involving the interaction of nonlinearity and dispersion effects. Then, it was used as a model for shock wave generation, solitons, turbulence, boundary layer behavior and mass transport in many fields such as fluid dynamics, plasma physics, aerodynamics and lattice dynamics, and many studies were performed on by mathematicians (Adem and Khalique, 2012; Bona and Smith, 1975; Kato, 1979; Kenig et al., 1991; S. Zhang et al., 2008; Zhou et al., 2003). Bakırtaş and Antar (2003), Bakırtaş and Demiray (2005), using the method of reductive perturbation, studied the weakly nonlinear propagation of waves in elastic tubes filled with non-compressible viscous fluid where long-wave approximation was used. The KdV equation obtained in (Bakırtaş and Antar, 2003) and (I. Bakırtaş and Demiray, 2005), by treating blood as a non-compressible viscous fluid and the arteries as tapered, flexible, thin-walled, long circular conical tube, is as follows

$$u_t + \nu_1 u u_x + \nu_2 u_{xxx} + \mu(t) u_x = 0, \tag{4.11.2}$$

where ν_1 and ν_2 are constants due to the initial deformation of the tube material, and $\mu(t)u_x$ represents the contribution of the tapering of tube. The equation (4.11.2) is called the extended-KdV equation by the authors of (Bakırtaş and Antar, 2003).

The deterministic KdV equation is insufficient in the modeling of physical phenomena that include uncertainty due to the difficulty of describing the physical systems of the real-world with deterministic equations. In order to eliminate this deficiency, Wadati (Wadati, 1983), added a forcing term comprising external noise

$$u_t - 6 u u_x + u_{xxx} = \eta(t), \tag{4.11.3}$$

and Iizuka (Iizuka, 1993) added a derivative term multiplicated by noise with long-range correlation

$$u_t + u u_x + u_{xxx} - \eta(t) u_x = 0. \tag{4.11.4}$$

Eq.(4.11.3) models traveling waves in noisy plasmas while Eq. (4.11.4) with multiplicative noise arises in the modeling of diffusive behavior of the solitons (especially anomalous diffusion of solitons (Iizuka, 1993). In (Wadati, 1983), Wadati discovered diffusion of soliton for Eq. (4.11.3). Moreover, Wadati and Akutsu (1984) considered the effect of friction by adding a damping term to (4.11.3). Lin et al. (2006) studied Eq. (4.11.4) with homogeneous boundary conditions. They obtained numerical solutions of (4.11.4) using discontinuous Galerkin and finite difference methods with considering three different noise types: additive noise, multiplicative noise, and a combination of both noises. The main purpose here is to obtain the periodic wave solutions of Eq. (4.11.1) with a random term of white noise type. As known, in nonlinear science construction of traveling wave solutions has an important role and several methods have been developed to obtain traveling wave solutions. Among these methods, we use the F-expansion method for obtaining the periodic solutions. We also aim to demonstrate the effect of noise on the wave-form by visualizing the solutions with different noise functions. To illustrate the F-expansion method and the possibilities it offers, we now investigate stochastic extended KdV Equation in detail. Depending on the steps presented in Section (3.2.3), we will present the detailed solution to the Wick-type stochastic extended KdV equation (4.11.1) as follows. By applying the Hermite transform to Eq. (4.11.1), we obtain the following equation

$$\widetilde{U_t} + \widetilde{H_1}(t,z)\widetilde{U_x} + \widetilde{H_2}(t,z)\widetilde{U}\widetilde{U_x} + \widetilde{H_3}(t,z)\widetilde{U_{xxx}} = 0, \tag{4.11.5}$$

where $z = (z_1, z_2, \dots) \in C_c^{\mathrm{N}}$ is a vector parameter. Let's denote $u(t,x,z) = \widetilde{U}(t,x,z)$ and $H_i(t,z) = \widetilde{H_i}(t,z)(i = 1,2,3)$ for simplicity. Suppose the formal solution to Eq. (4.11.5) takes the following form

$$u = u(\zeta), \quad \zeta = f(t,x)x + g(t,x). \tag{4.11.6}$$

Whereas in the previous equation, both $f(t,x)$ and $g(t,x)$ are functions that will be determined later. Anyway, let's consider the solution of the Eq. (4.11.5). It is expressed in the following form

$$u(\zeta) = \sum_{i=0}^{n} a_i(t,z)F^i(\zeta). \tag{4.11.7}$$

The balancing integer term n can be calculated using the principle of homogeneous balance between the highest order linear term and the nonlinear term in Eq. (4.11.9). Also the functions $a_i(t,z)$ and $F^i(\zeta)(i = 0,1,2,\dots,n)$ will be determined later. Assuming that solutions of the elliptic equation (4.11.8) is the function $F(\zeta)$ given in the equation (4.11.7).

$$F_\zeta^2 = A_1 + A_2F^2(\zeta) + A_3F^4(\zeta), \tag{4.11.8}$$

where A_1, A_2, A_3 and $F(\zeta)$ are values determined using the corresponding values which geves in Table 3.3 .The following equation is obtained by substituting Eq.(4.11.6) into Eq. (4.11.5).

$$(f_t x + g_t)u_\zeta + H_1 f u_\zeta + H_2 f u u_\zeta + H_3 f^3 u_{\zeta\zeta\zeta} = 0. \tag{4.11.9}$$

Balancing uu_ζ with $u_{\zeta\zeta\zeta}$ gives $n = 2$, then, the ansatz takes the following form

$$u = a_0 + a_1F + a_2F^2. \tag{4.11.10}$$

Now from Eq.(4.11.10) and Eq. (4.11.9) we get

$$a_{0t} + a_{1t}F + a_{2t}F^2 + 2H_3a_2f^3FF_{\zeta\zeta\zeta} + H_3f^3a_1F_{\zeta\zeta\zeta} + 6H_3a_2f^3F_{\zeta\zeta}F_\zeta + 2H_2a_2^2fF^3F_\zeta + 3H_2a_1a_2fF^2F_\zeta$$
$$+(2a_2f_tx + 2H_2xa_2f + 2H_1a_2f + H_2a_1^2f + 2a_2g_t + 2H_2a_0a_2f)FF_\zeta$$
$$+(a_1f_tx + a_1g_t + H_1a_1f + H_2a_0a_1f)F_\zeta = 0. \quad (4.11.11)$$

According to Eq. (4.11.8), we get

$$F_{\zeta\zeta} = A_2F + 2A_3F^3, \quad (4.11.12)$$

$$F_{\zeta\zeta\zeta} = (A_2 + 6A_3F^2)F_\zeta. \quad (4.11.13)$$

Using Eqs. (4.11.8),(4.11.12) and (4.11.13) into Eq. (4.11.11), we

$$a_{0t} + a_{1t}F + a_{2t}F^2 + (a_1(f_tx + g_t) + H_1a_1f + H_2a_0a_1f + H_3a_1f^3A_2)F_\zeta$$
$$+(2a_2(f_tx + g_t)+2H_1a_2f + H_2a_1^2f + 2H_2a_0a_2f + 8H_3a_2f^3A_2)FF_\zeta$$
$$+(3H_2a_1a_2f + 6H_3a_1f^3A_3)F^2F_\zeta + (2H_2a_2^2f + 24H_3a_2f^3A_3)F^3F_\zeta = 0. \quad (4.11.14)$$

From the Eq. (4.11.14), we get the following system of algebraic equations for a_0, a_1, a_2, f and g after collecting the coefficients of $F^i, F^iF_\zeta (i = 1,2,3)$ with each other and equating them to zero, we have

$$a_{0t} = a_{1t} = a_{2t} = 0, \quad (4.11.15)$$

$$a_1(f_tx + g_t + H_1f + H_2a_0f + H_3f^3A_2) = 0, \quad (4.11.16)$$

$$2a_2(f_tx + g_t + H_1f) + H_2f(a_1^2 + 2a_0a_2) + 8H_3a_2f^3A_2 = 0, \quad (4.11.17)$$

$$3a_1f(H_2a_2 + 2H_3f^2A_3) = 0, \quad (4.11.18)$$

$$2a_2f(H_2a_2 + 12H_3f^2A_3) = 0. \quad (4.11.19)$$

Solving Eqs. (4.11.15),(4.11.16), we get

$$a_0 = c_0, \quad a_1 = c_1, \quad a_2 = c_2, \quad (4.11.20)$$

where c_0, c_1 and c_2 are arbitrary constants. Through equations (4.11.16) and (4.11.17), we find that

$$f(t, z) = f_0, \quad (4.11.21)$$

where $f_0 \neq 0$ is constant. Also from both equations (4.11.19) and (4.11.21), we derive

$$a_2(t, z) = -\frac{12A_3f_0^2H_3(t, z)}{H_2(t, z)}. \quad (4.11.22)$$

Comparing the equations (4.11.20) and (4.11.22), yields

$$H_3(t,z) = \gamma H_2(t,z), \tag{4.11.23}$$

where γ is constant. From both equations (4.11.23) and (4.11.22), we have

$$a_2(t,z) = -12A_3 f_0^2 \gamma. \tag{4.11.24}$$

Also, through equations (4.11.18) and (4.11.20), we find that

$$a_1(t,z) = c_1 = 0. \tag{4.11.25}$$

Now, by using equations (4.11.21), (4.11.24) and (4.11.25) in equation (4.11.17), we get

$$g(t,z) = -f_0 \int_0^t H_1(s,z)ds - (f_0 a_0 + 4\gamma f_0^3 A_2) \int_0^t H_2(s,z)ds\,. \tag{4.11.26}$$

In view of Eqs. (4.11.6), (4.11.21) and (4.11.26) we get

$$\zeta = f_0 x - f_0 \int_0^t H_1(s,z)ds - (f_0 a_0 + 4\gamma f_0^3 A_2) \int_0^t H_2(s,z)ds\,. \tag{4.11.27}$$

In view of equations (4.11.10), (4.11.20), (4.11.22) and (4.11.25), we get the solution of the Eq. (4.11.5) as follows

$$u(t,x,\zeta) = c_0 - 12A_3 f_0^2 \gamma F(\zeta)^2, \tag{4.11.28}$$

where ζ is calculated based on Eq.(4.11.27) and $F(\zeta)$ is all solutions of the Jacobian elliptic function fulfilled for Eq. (4.11.8). Let $h(t)$ be integrable function on R_- and $b_i (i = 1,2)$ be arbitrary constants and

$$H_1(t) = b_1 W(t), \quad H_2(t) = h(t) + b_2 W(t). \tag{4.11.29}$$

In Eq. (4.11.29), Gaussian white noise and Brown motion are denoted by $W(t)$ and $B(t)$, respectively. Also from the stochastic analysis, we have $W(t) = \dot{B}(t)$. Through Eq. (4.11.23), we have

$$\begin{aligned} H_1(t) &= b_1 W(t), \\ H_2(t) &= h(t) + b_2 W(t), \\ H_3(t) &= \gamma\big(h(t) + b_2 W(t)\big). \end{aligned} \tag{4.11.30}$$

Using the Hermite transformations for Eqs. (4.11.29) and (4.11.30) , respectively, we obtain

$$\widetilde{H_1}(t,z) = b_1\widetilde{W}(t,z),$$
$$\widetilde{H_2}(t,z) = h(t) + b_2\widetilde{W}(t,z),$$
$$\widetilde{H_3}(t,z) = \gamma\left(h(t) + b_2\widetilde{W}(t,z)\right), \tag{4.11.31}$$

where $\widetilde{W}(t,z) = \sum_{k=1}^{\infty}\int_0^t \eta_k(s)\mathrm{d}\, s\, z_k$. Through Eqs.(4.11.28) and (4.11.27), we can find the general formal

$$U(t,x) = c_0 - 12A_3 f_0^2\gamma F^{2\diamond}(\bar{\zeta}), \tag{4.11.32}$$

where $F(\bar{\zeta})$ is all solutions of the Jacobian elliptic function fulfilled for Eq.(4.11.8) and

$$\bar{\zeta} = f_0 x - f_0\int_0^t H_1(s)ds - (f_0 a_0 + 4\gamma f_0^3 A_2)\int_0^t H_2(s)ds$$
$$= f_0 x - f_0 b_1 B(t) - (f_0 a_0 + 4\gamma f_0^3 A_2)\left(b_2 B(t) + \int_0^t h(s)ds\right). \tag{4.11.33}$$

In view of $\exp^{\diamond}\left(B(t)\right) = \exp\left(B(t) - \frac{1}{2}t^2\right)$ (see (Holden et al., 1996), Lemma 2.6.16). By means of Eqs.(4.11.33), (4.11.32), (4.11.30) and (4.11.31) we obtain

$$U(t,x) = c_0 - 12A_3 f_0^2\gamma F^2(\zeta), \tag{4.11.34}$$

where

$$\zeta = f_0 x - f_0\int_0^t b_1\,\delta B(s) - (f_0 a_0 + 4\gamma f_0^3 A_2)\int_0^t \left(h(s)ds + b_2\delta B(s)\right)$$
$$= f_0 x - f_0 b_1\left(B(t) - \frac{1}{2}t^2\right) - (f_0 a_0 + 4\gamma f_0^3 A_2)\left(b_2\left(B(t) - \frac{1}{2}t^2\right) + \int_0^t h(s)ds\right) \tag{4.11.35}$$

Also, it must be noted that we used the following relation in Eq.(4.11.35)

$$\int_R \Psi(t)\diamond W(t)\mathrm{d}t = \int_R \Psi(t)\delta B(t), \quad \Psi(t) \in L^2(R), \tag{4.11.36}$$

where the stochastic integral $\int(.)\delta B(s)$ is the Skorohod integral. Using the values of $A_2, A_3, F(\zeta)$ from Table 3.3 in Eqs. (4.11.35) and (4.11.34), we obtain a series of solutions of Jacobian elliptic function of Eq. (4.11.1).

In view of case 1. as an example, we obtain

$$U_1(t,x) = c_0 - 12m^2 f_0^2 \gamma sn^2(\zeta), \qquad (4.11.37)$$

where

$$\zeta = f_0 x - f_0 b_1 \left(B(t) - \frac{1}{2}t^2\right) - \left(f_0 a_0 + 4\gamma f_0^3(1+m^2)\right)\left(b_2\left(B(t) - \frac{1}{2}t^2\right) + \int_0^t h(s)ds\right) \qquad (4.11.38)$$

Also, we know that $sn(\zeta) \to \tanh(\zeta)$ when $m \to 1$. Therefore, we can find a stochastic soliton-like solution for Eq. (4.11.1) in the following form:

$$U_1^*(t,x) = c_0 - 12 f_0^2 \gamma \tanh^2(\zeta), \qquad (4.11.39)$$

where

$$\zeta = f_0 x - f_0 b_1 \left(B(t) - \frac{1}{2}t^2\right) - (f_0 a_0 + 8\gamma f_0^3)\left(b_2\left(B(t) - \frac{1}{2}t^2\right) + \int_0^t h(s)ds\right) \qquad (4.11.40)$$

Graphs of some solutions for particular values of parameters $c_0, f_0, b_1, b_2, a_0, \gamma$ and different values of, $B(t)$, $h(s)$ are visualized below.

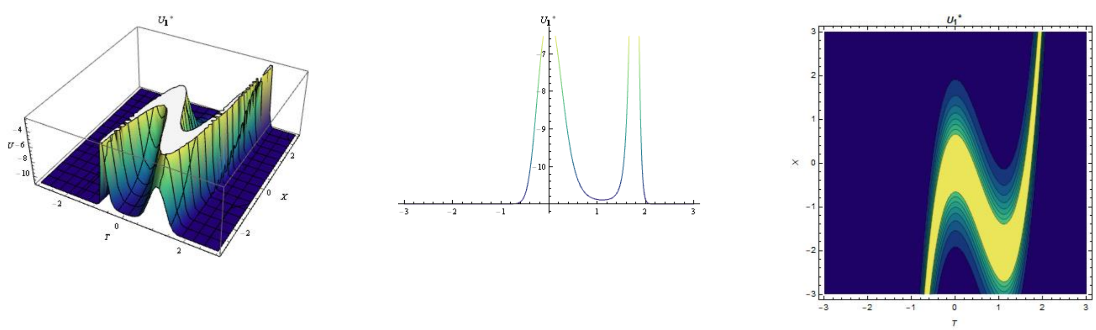

Figure 4.56. Graph of solution (4.11.39) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = 0, h(s) = s$

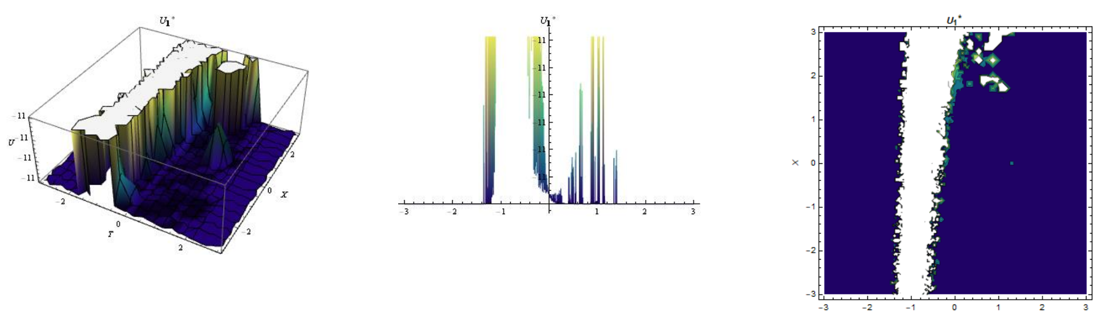

Figure 4.57. Graph of solution (4.11.39) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = e^{noise*t}$, $h(s) = s^2$

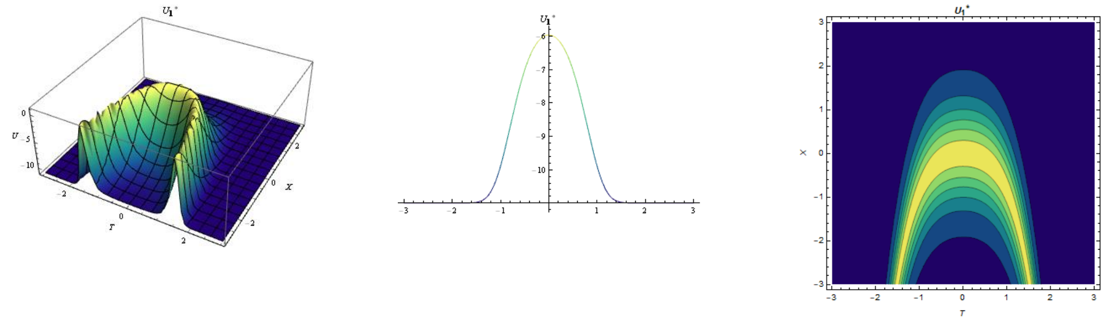

Figure 4.58. Graph of solution (4.11.39) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = 0$, $h(s) = \sin(s)$

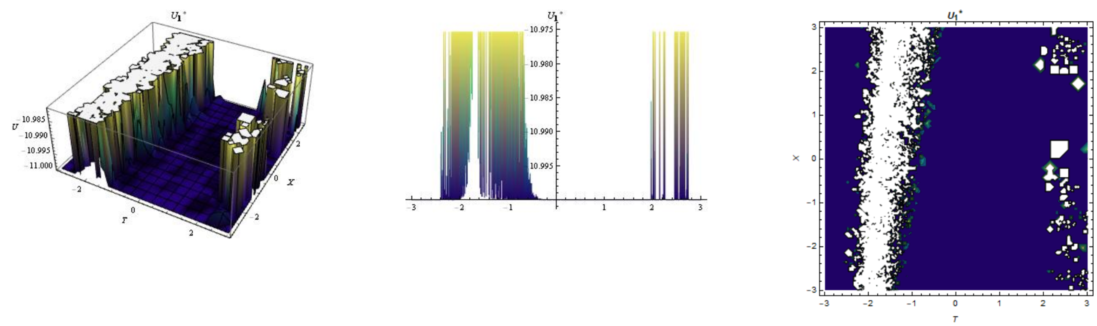

Figure 4.59. Graph of solution (4.11.39) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = e^{noise*t}$, $h(s) = sin(s)$

In view of case 3. We have

$$U_3(t,x) = c_0 + 12m^2 f_0^2 \gamma cn^2(\zeta), \tag{4.11.41}$$

where

$$\zeta = f_0 x - f_0 b_1 \left(B(t) - \frac{1}{2}t^2\right) - \left(f_0 a_0 - 4\gamma f_0^3 (1-2m^2)\right)\left(b_2\left(B(t) - \frac{1}{2}t^2\right) + \int_0^t h(s)ds\right) \tag{4.11.42}$$

Also, we known that $cn(\zeta) \to sech(\zeta)$ when $m \to 1$. So if that we can find a stochastic soliton-like solution for Equation (4.11.1) in the following form:

$$U_3^*(t,x) = c_0 + 12 f_0^2 \gamma \, \mathrm{Sech}^2(\zeta)\,, \tag{4.11.43}$$

with

$$\zeta = f_0 x - f_0 b_1 \left(B(t) - \frac{1}{2}t^2\right) - (f_0 a_0 + 4\gamma f_0^3)\left(b_2\left(B(t) - \frac{1}{2}t^2\right) + \int_0^t h(s)ds\right) \tag{4.11.44}$$

Graphs of some solutions for particular values of parameters c_0, f_0, b_1, b_2, a_0, γ and different values of, $B(t)$, $h(s)$ are visualized below.

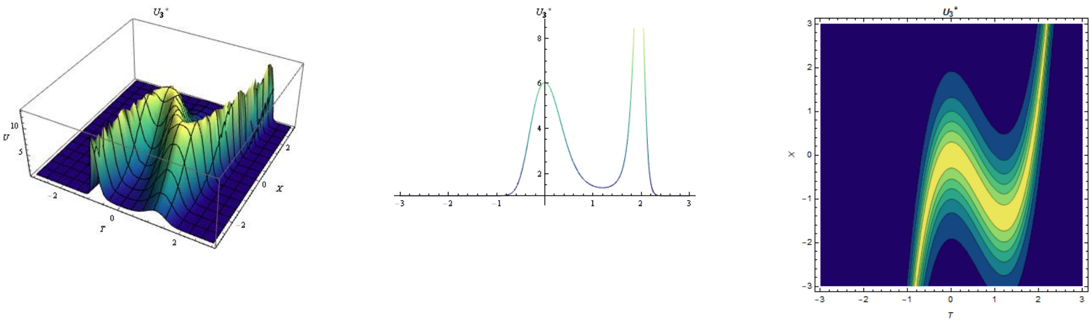

Figure 4.60. Graph of solution (4.11.43) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = 0$, $h(s) = s^2$

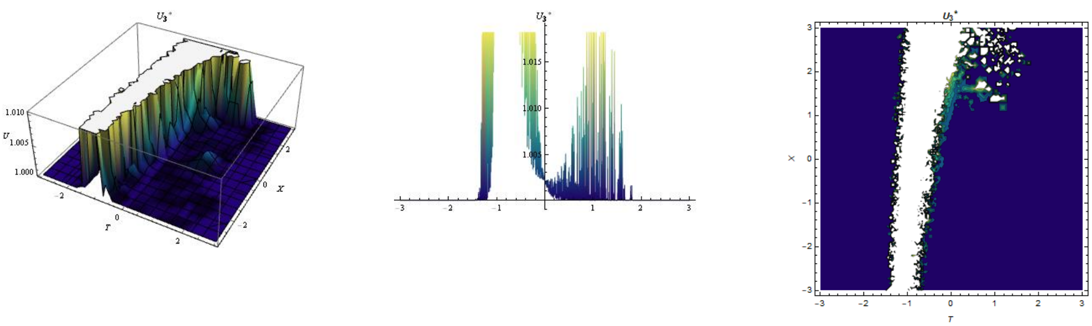

Figure 4.61. Graph of solution (4.11.43) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = e^{noise*t}$, $h(s) = s^2$

In view of case 14. we obtain

$$U_{14}(t,x) = c_0 - 3m^2 f_0^2 \gamma \big(sn(\zeta) \pm icn(\zeta)\big)^2, \tag{4.11.45}$$

where

$$\zeta = f_0 x - f_0 b_1 \left(B(t) - \frac{1}{2}t^2\right) - \left(f_0 a_0 - 2\gamma f_0^3 (m^2 - 2)\right)\left(b_2\left(B(t) - \frac{1}{2}t^2\right) + \int_0^t h(s)ds\right). \tag{4.11.46}$$

Also, we known that $sn(\zeta) \to \tanh(\zeta)$ and $cn(\zeta) \to sech(\zeta)$ when $m \to 1$. So, if that we can find a stochastic soliton-like solution for Equation (4.11.1) in the following form:

$$U_{14}^*(t,x) = c_0 - 3f_0^2 \gamma (tanh\,(\zeta) \pm iSech\,(\zeta))^2, \tag{4.11.47}$$

with

$$\zeta = f_0 x - f_0 b_1 \left(B(t) - \frac{1}{2}t^2\right) - (f_0 a_0 + \gamma f_0^3)\left(b_2\left(B(t) - \frac{1}{2}t^2\right) + \int_0^t h(s)ds\right), \tag{4.11.48}$$

Graphs of some solutions for particular values of parameters c_0, f_0, b_1, b_2, a_0, γ and different values of, $B(t)$, $h(s)$ are visualized below.

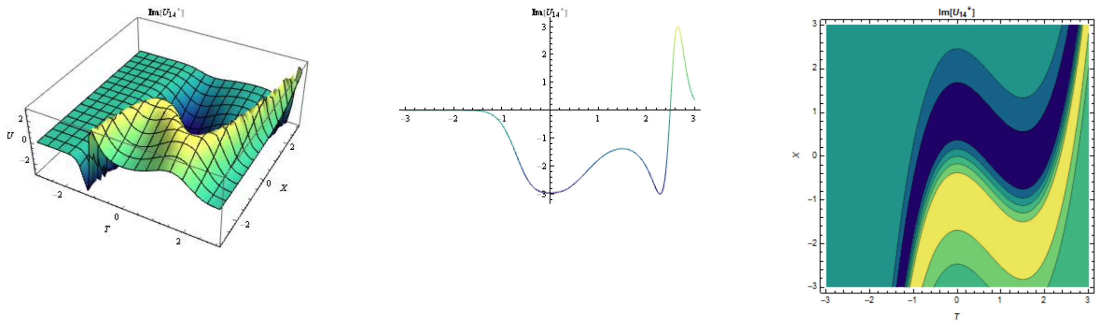

Figure 4.62. Graph of solution (4.11.47) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = 0$, $h(s) = s^2$, $Im[U_{14}^{*}(x,t)]$

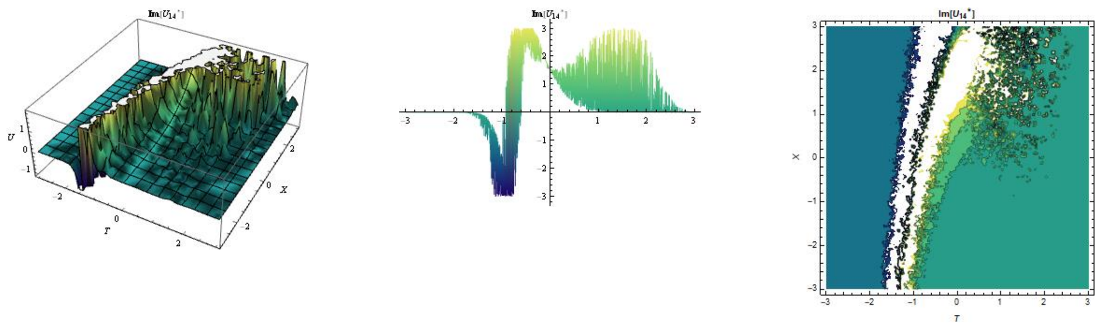

Figure 4.63. Graph of solution (4.11.47) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = e^{noise*t}$, $h(s) = s^2$, $Im[U_{14}^{*}(x,t)]$

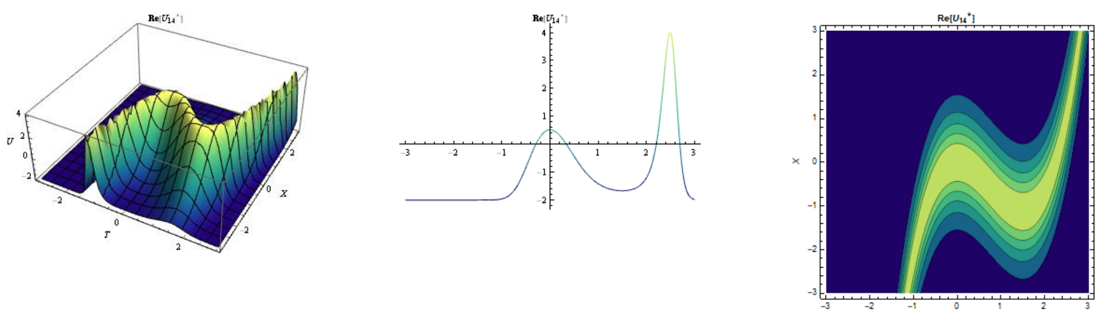

Figure 4.64. Graph of solution (4.11.47) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = 0$, $h(s) = s^2$, $Re[U_{14}^{*}(x,t)]$

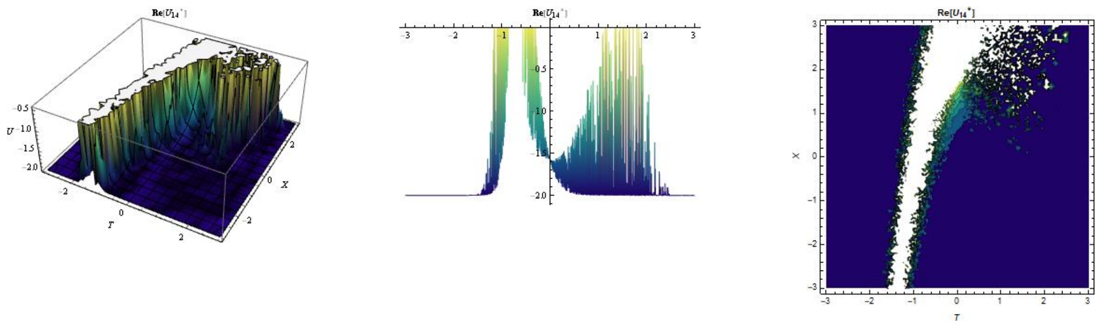

Figure 4.65. Graph of solution (4.11.47) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = e^{noise*t}$, $h(s) = s^2$, $Re[U_{14}^*(x,t)]$

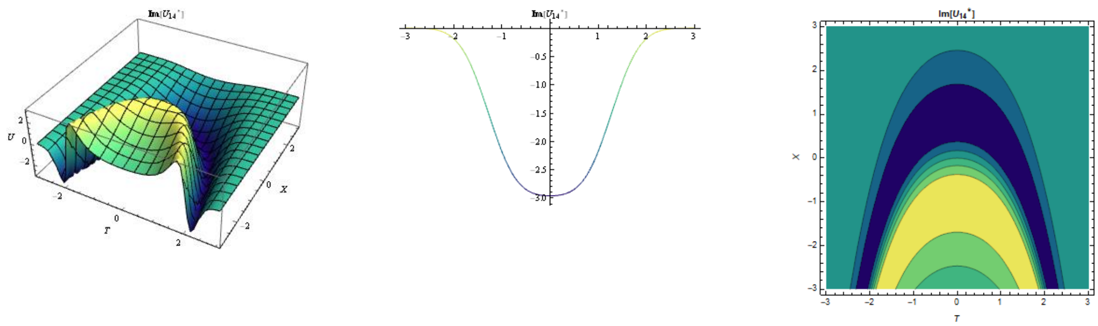

Figure 4.66. Graph of solution (4.11.47) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = 0$, $h(s) = \sin(s)$, $Im[U_{14}^*(x,t)]$

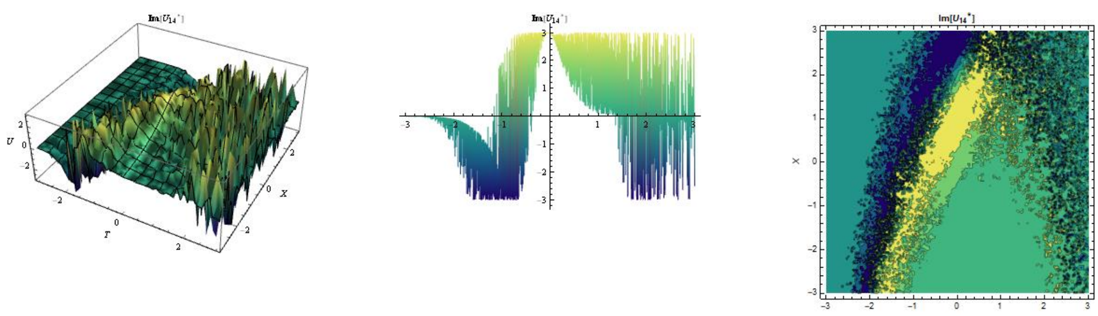

Figure 4.67. Graph of solution (4.11.47) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = sin(noise * t)$, $h(s) = sin(\mathrm{s})$, $Im[U_{14}^*(x,t)]$

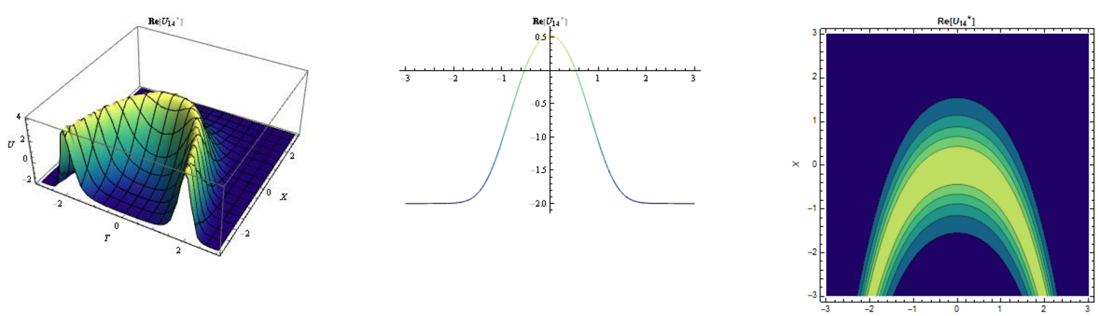

Figure 4.68. Graph of solution (4.11.47) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = 0$, $h(s) = sin(s)$, $\mathrm{Re}[U_{14}^*(x,t)]$

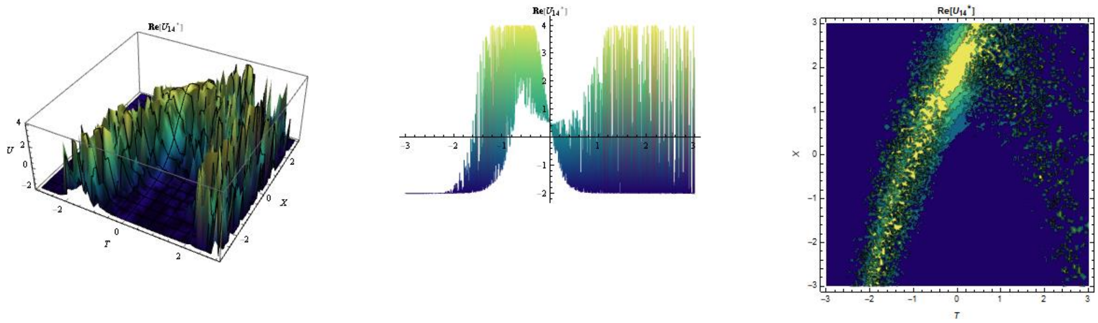

Figure 4.69. Graph of solution (4.11.47) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = sin(noise * t)$, $h(s) = sin(s)$, $\mathrm{Re}[U_{14}^{*}(x,t)]$

In view of case 22. we derive

$$U_{22}(t,x) = c_0 + 3f_0^2\gamma\big(mcn(\zeta) \pm dn(\zeta)\big)^2, \tag{4.11.49}$$

where

$$\zeta = f_0x - f_0b_1\left(B(t) - \frac{1}{2}t^2\right) - \left(f_0a_0 + 2\gamma f_0^3(1+m^2)\right)\left(b_2\left(B(t) - \frac{1}{2}t^2\right) + \int_0^t h(s)ds\right). \tag{4.11.50}$$

Also, we known that $dn(\zeta) \to sech(\zeta)$ and $cn(\zeta) \to sech(\zeta)$. So, if that we can find a stochastic soliton-like solution for Equation (4.11.1) in the following form.

$$U_{22}^{*}(t,x) = c_0 + 3f_0^2\gamma\big(Sech(\zeta) \pm Sech(\zeta)\big)^2, \tag{4.11.51}$$

with

$$\zeta = f_0x - f_0b_1\left(B(t) - \frac{1}{2}t^2\right) - (f_0a_0 + 4\gamma f_0^3)\left(b_2\left(B(t) - \frac{1}{2}t^2\right) + \int_0^t h(s)ds\right) \tag{4.11.52}$$

Graphs of some solutions for particular values of parameters $c_0, f_0, b_1, b_2, a_0, \gamma$ and different values of, $B(t)$, $h(s)$ are visualized below.

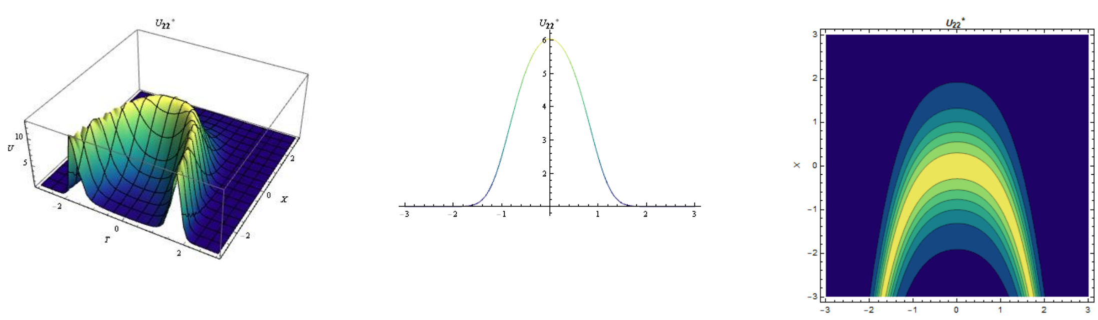

Figure 4.70. Graph of solution (4.11.51) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = 0$, $h(s) = \sin(s)$

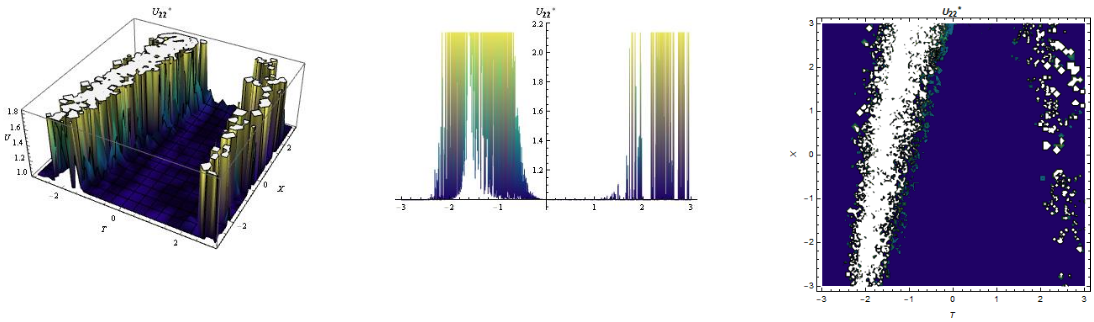

Figure 4.71. Graph of solution (4.11.51) for $c_0 = f_0 = b_1 = b_2 = a_0 = \gamma = 1$, $B(t) = e^{noise*t}$, $h(s) = \sin(s)$

By the following properties of the Jacobian elliptic function

$$\lim_{m\to 1} sn\,(\zeta) = \tanh(\zeta)\,, \quad \lim_{m\to 1} cn\,(\zeta) = sech(\zeta), \quad \lim_{m\to 1} dn\,(\zeta) = sech(\zeta) \tag{4.11.53}$$

$$\lim_{m\to 1} ns\,(\zeta) = coth(\zeta), \quad \lim_{m\to 1} cs\,(\zeta) = csch(\zeta), \quad \lim_{m\to 1} ds\,(\zeta) = cech(\zeta) \tag{4.11.54}$$

$$sd(\zeta) = \frac{sn(\zeta)}{dn(\zeta)}, cd(\zeta) = \frac{cn(\zeta)}{dn(\zeta)}, \quad nd(\zeta) = \frac{1}{dn(\zeta)} \tag{4.11.55}$$

$$sc(\zeta) = \frac{sn(\zeta)}{cn(\zeta)}, dc(\zeta) = \frac{dn(\zeta)}{cn(\zeta)}, \quad nc(\zeta) = \frac{1}{cn(\zeta)} \tag{4.11.56}$$

The solutions of the hyperbolic function corresponding to Eq. (4.11.1) We can derive it simply from solutions of the Jacobian elliptical function not mentioned here

5. DISCUSSION AND CONCLUSION

In this thesis, we investigated the effect of noise on the solutions of stochastic evolution equations. For this purpose, several types of evolution equations have been treated, and two different analytical methods were employed. Solutions of stochastic KdV-Burgers, stochatic KdV, stochastic Burgers, stochastic Kuramoto-Sivashinsky and stochastic Kawahara equation are obtained by means of Galilean transformation and tanh, extended tanh methods. Solutions of a stochastic Wick-type extended-KdV equation are found by Hermite transform and by means of Jacobi elliptic functions.

As soon as we are aware, the stochastic KdV-Burgers equation has not been dealed analyticaly before. The solitary wave solution obtained for KdV-Burgers equation

$$\begin{aligned} u_3(x,t) &= -\frac{3B^2}{25R}(1+\tanh(-\xi))(\tanh(-\xi)-3) \\ &= \frac{3B^2}{25R}(1-\tanh\xi)(3-\tanh\xi), \end{aligned} \tag{5.1}$$

can also be written in the following form:

$$\begin{aligned} F(Y) &= \frac{3B^2}{25R}(1-M)(1+M)+\frac{6B^2}{25R}(1-M) \\ &= \frac{3B^2}{25R}sech^2\xi+\frac{6B^2}{25R}(1-\tanh\xi), \end{aligned} \tag{5.2}$$

where $M = \tanh\left[x-\frac{B}{10R^2}\left(\frac{6B^2}{25R}t\right)\right]$, and $\xi = x-\frac{B}{10R^2}\left(\frac{6B^2}{25R}t\right)$. It represents a particular combination of a solitary wave [first term on the r.h.s. of (5.2)] with a shock-wave (second term) due to the presence of $-BU_{XX}$ in Eq. (4.1.1). One can see from the graphs that for the deterministic cases, i.e. for $W(T)=0$, (for example Figure 4.1 , Figure 4.5), the graphs are smooth which shows that the wave-form does not change. But for the stochastic cases (i.e. when $W(T)\neq 0$,), the impact of the noise can be seen clearly from the graphs (for example Figure 4.2, Figure 4.3, Figure 4.6 and Figure 4.12) which indicates that wave-form changes under the effect of an external noise.

We also investigated the analytical solutions of stochastic Kuramoto-Sivashinsky and Kawahara equations by transforming them into the deterministic counterparts by using Galilean transformation. Afterward, we used the tanh-function method for obtaining the soliton solutions of the deterministic counterpart of the stochastic ones. We also obtained periodic solutions for stochastic Kawahara equation. We visualize some of their solutions with and without noise to compare the effect of the noise. One can see from the graphs that different noise functions changes the wave-form during the propagation of the solitons.

The analytical solutions of a Wick type stochastic extended KdV equation arising in the modeling of the flow of blood in the arteries are studied by means of Hermite transform and F-expansion method.

The Galilean and Hermite transformations are very useful tools in finding deterministic equivalents of stochastic equations, and they may be used to convert some other stochastic evolution equations arising in different fields such as physics, finance into their deterministic counterparts.

If we add a singular perturbation to the previously studied equations in this work, we will obtain stochastic singularly perturbed equations, so that in this case the nonlinear analytical methods do not give solutions and therefore we need numerical methods to find solutions to these equations, for example as the finite difference methods for more details see (Sakar et al., 2019) .

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EXTENDED TURKISH SUMMARY

ÖZ

Doğrusal olmayan evolüsyon denklemler, t zaman değişkenini bir bağımsız değişken olarak içeren ve sadece matematiğin birçok alanında değil fizik, mekanik ve materyal bilimi gibi diğer bilim dallarında da ortaya çıkan kısmi diferansiyel denklemlerdir. Navier-Stokes ve Euler denklemleri akışkanlar mekaniğinde, reaksiyon-difüzyon denklemleri ısı transferlerinde ve biyolojik bilimlerde, Klein-Gordon ve Schrödinger denklemleri kuantum mekaniğinde, Cahn-Hilliard denklemi ise materyal biliminde ortaya çıkan lineer olmayan evolüsyon denklemlerinden sadece birkaçıdır. Deterministik modeller genellikle birçok küçük pertürbasyonun etkisini ihmal ettiğinden stokastik denklemler olaylara daha iyi uyum sağlamaktadır. Örneğin, sığ suların yüzeyindeki dalgalar modellenirken, sıvı yüzeyini etkileyen sabit olmayan bir basınç veya tabakanın tabanının düz olmadığı durumda gerçekçi bir model oluşturulabilmesi için bu etkileri içeren stokastik bir terimin denkleme eklenmesi denklemi daha anlamlı kılacaktır. Bu tezde bu terimlerin, yani gürültünün bazı evolüsyon denklemlerinin çözümleri üzerindeki etkisi araştırıldı. Doğrusal olmayan stokastik evolüsyon denklemler fizik, kimya, biyoloji, ekonomi ve finans alanlarında çeşitli açılardan geniş bir uygulama alanına sahiptir. Doğrusal olmayan evolüsyon denklemlerin tam çözümlerinin bulunması, denklemlerin modellediği fiziksel veya mekaniksel problemler açısından olduğu kadar, kullanılan nümerik yöntemlerin doğruluğunun test edilmesi açısından da oldukça önemlidir. Bu önem stokastik evolüsyon denklemler için de geçerlidir.

Anahtar kelimeler: Stochastic evolüsyon denklemler, KdV-Tipli denklemler, Hermite dönüşümü, Galilean dönüşümü, Jacobi eliptik fonksiyonlar.

1. MATERYAL VE YÖNTEM

Tam çözümlerin bulunması için birçok yöntem geliştirilmiştir. Bu çalışmada, bu yöntemlerden tanh, extended tanh ve F-açılım metotları kullanılmıştır. Bu yöntemlerin kullanılabilmesi için, çalışılan stokastik denklemlerin deterministik karşılıklarını elde etmek amacıyla Hermite dönüşümü ve Galilean dönüşümü kullanıldı ve daha sonra yukarıda bahsedilen yöntemlerle çözümler elde edildi. Bu tez yedi bölümden oluşmaktadır. İlk bölüm giriş niteliğinde olup, stokastik diferansiyel denklemlere neden ihtiyaç duyulduğu bir örnekle anlatılmıştır. İkinci bölüm çalışılan denklemlerle ilgili literatürde yapılmış çalışmaları ve elde edilen sonuçlardan bazılarını içermektedir. Üçüncü bölüm, tezi daha anlaşılır olması için gerekli tanım, teorem ve kavramları içermekte, dördüncü bölüm ise kullanılan yöntem-leri detaylı olarak anlatmaktadır. Tezin beşinci bölümünde, Galilean dönüşümü ve tanh, genişletilmiş tanh yöntemleri kullanılarak stokastik KdV-Burgers, stokastik KdV, stokastik Burgers, stokastik Kuramoto- Sivashinsky ve stokastik Kawahara denklemlerinin analitik çözümleri elde edilmiştir. Tezin altıncı bölümünde, stokastik Wick-tipi genişletilmiş KdV denkleminin çözümleri Hermite dönüşümü ve Jacobi eliptik fonksi-yonları kullanılarak bulunmuştur. Tezin yedinci ve son bölümünde ise elde edilen sonuçlar özetlenmiş ve gelecekte yapılabilecek çalışmalar anlatılmıştır. Gürültünün etkisinin daha iyi görülebilmesi için bazı çözümlerin grafiklerine yer verilmiştir.

2. BULGULAR VE TARTIŞMA

Beşinci bölümün ilk problemi stochastic KdV-Burgers denklemi

$$U_t + UU_X - BU_{XX} + RU_{XXX} = \eta(T), \tag{7.1}$$

çözümlerinin bulunmasıdır. Burada $\eta(T)$ terimi dış gürültüyü temsil etmektedir. Bu denklemin deterministik karşılığını bulmak için aşağıdaki Galilean dönüşümü kullanılmıştır:

$$U(X,T) = u(x,t) + W(T), x = X + m(t), t = T, \tag{7.2}$$

$$m(T) = -\int_0^T W(T')d\,T', \quad W(T) = \int_0^T \eta(T')d\,T'. \tag{7.3}$$

Bu dönüşüm sonrasında (7.3) denklemi

$$u_t + uu_x - Bu_{xx} + Ru_{xxx} = 0, \tag{7.4}$$

denklemine dönüşür. Bu denklem için iki farklı durum incelenmiştir:

1. Denklem

$$z \to \pm\infty \quad iken \quad V(z) \to 0, \quad \frac{d^nV(z)}{dz^n} \to 0, \quad (n = 1,2,\dots), \tag{7.5}$$

   sınır koşullarıyla beraber çalışılmış,

2. Denklem herhangi bir sınır koşulu olmadan çalışılmıştır.

Birinci durum için elde edilen çözümlerden biri

$$U_1(X,T) = -\frac{3B^2}{25R}(1+\tanh[\phi_1(X,T)])^2 + W(T), \tag{7.6}$$

diğeri ise,

$$U_3(X,T) = -\frac{3B^2}{25R}(1+\tanh[\phi_2(X,T)])(\tanh[\phi_2(X,T)] - 3) + W(T), \tag{7.7}$$

formundadır. Burada $\phi_1(X,T) = \left[\frac{B}{10R}\left(\frac{6B^2}{25R}T + X - \int_0^T W(T')dT'\right)\right]$ ve $\phi_2(X,T) = \left[\frac{B}{10R^2}\left(\frac{6B^2}{25R}T - X + \int_0^T W(T')dT'\right)\right]$ dir. (7.7) çözümü aynı zamanda

$$\begin{aligned} F(Y) &= \frac{3B^2}{25R}(1-M)(1+M) + \frac{6B^2}{25R}(1-M) + W(T) \\ &= \frac{3B^2}{25R}sech^2[\phi_2(X,T)] + \frac{6B^2}{25R}(1-\tanh[\phi_2(X,T)]) + W(T), \end{aligned} \tag{7.8}$$

formunda da yazılabilir. Burada $M = \tanh[\phi_2(X,T)]$ dir. (7.8) denklemine bakıldığında, sağ taraftaki ilk terim soliter dalgayı, ikinci terim şok dalgasını, üçüncü terim ise, dış gürültüyü temsil eder. Gürültü teriminin etkisinin daha iyi görülebilmesi için çözüm farklı fonksiyonlar alınarak görselleştirilmiştir.

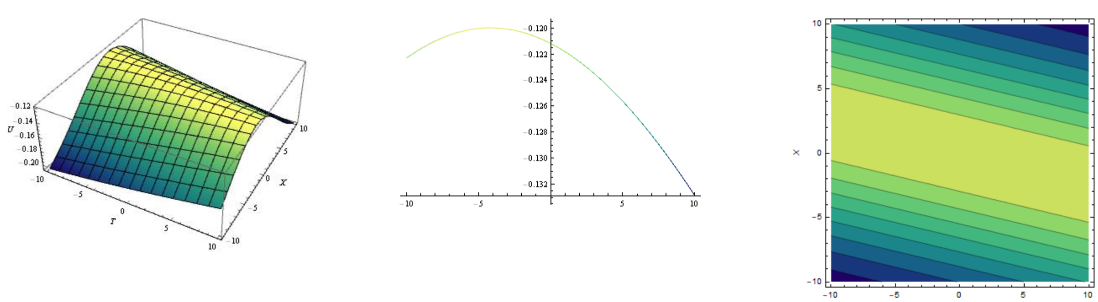

Figure 0.1. (7.6) çözümünün $B = R = 1$ ve $W(T) = 0$ alınarak elde edilen 3D, 2D ve kontur grafikleri.

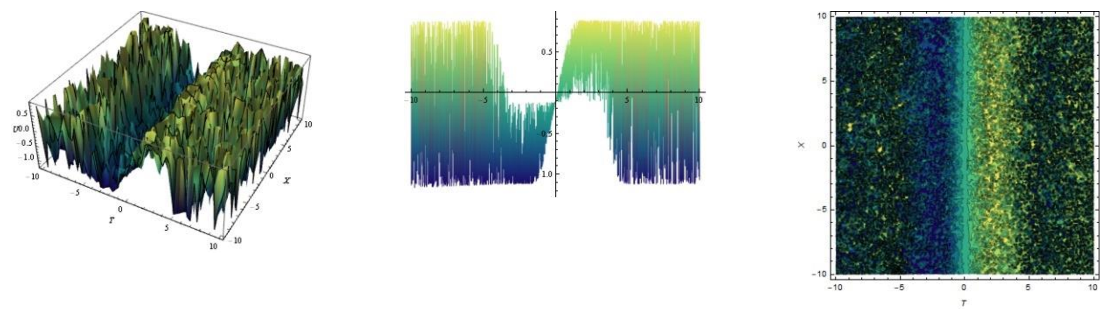

Figure 0.2. (7.6) çözümünün $B = R = 1$ ve $W(T) = \sin[noise * T]$ alınarak elde edilen 3D, 2D ve kontur grafikleri.

Yukarıdaki grafiklerden görüldüğü gibi $W(T) = 0$ olması durumunda dalga formu düzgün bir yapıya sahip, ancak $W(T) = \sin[noise * T]$ durumunda bu düzgünlük yerini zigzaglı, düzensiz bir yapıya bırakmıştır. İkinci durumda yani, herhangi bir sınır koşulu alınmadan elde edilen çözüm

$$U(X,T) = a_0 - \frac{3B^2}{25R}(1 - sech^2[\phi_3(X,T)] - 2\tanh[\phi_3(X,T)]) + W(T), \tag{7.9}$$

formundadır. Burada $\phi_3(X,T) = \left[\frac{B}{10R}\left(\left(\frac{3B^2}{25R} - a_0\right)T + X - \int_0^T W(T')dT'\right)\right]$ dir.

Extended tanh metodu ve sınır koşulları kullanılarak

$$U_1(X,T) = \frac{3B^2}{25R}\coth^2[\phi_4(X,T)]\,(\tanh[\phi_4(X,T)] + 1)(3\tanh[\phi_1(X,T)] - 1) + W(T), \tag{7.10}$$

$$U_2(X,T) = -\frac{3B^2}{25R}\coth^2[\phi_5(X,T)]\,(\tanh[\phi_5(X,T)] + 1)^2 + W(T) \tag{7.11}$$

çözümleri elde edilir. Burada $\phi_4(X,T) = \left[\frac{B}{10R}\left(\frac{6B^2}{25R}T - X + \int_0^T W(T')dT'\right)\right]$ ve

$\phi_5(X,T) = \left[\frac{B}{10R}\left(\frac{6B^2}{25R}T + X - \int_0^T W(T')dT'\right)\right]$ dir. Bu çözümler dışında iki ayrı çözüm daha bulunmuştur. Sınır koşulları kullanılmazsa

$$U(X,T) = a_0 + \frac{6B^2}{50R}\left(\coth^2[\phi_6(X,T)] + 2\coth[\phi_6(X,T)] - \frac{1}{2}\right) + W(T) \tag{7.12}$$

çözümü elde edilir. Burada $\phi_6(X,T) = \left[\frac{B}{10R}\left(\left(a_0 - \frac{3B^2}{50R}\right)T - X + \int_0^T W(T')dT'\right)\right]$ dir.

Stokastik KdV denklemi

$$U(X,T) = u(x,t) + W(T), x = X + m(t), t = T, \tag{7.13}$$

$$m(T) = -\int_0^T W(T')dT', \quad W(T) = \int_0^T \eta(T')dT' \tag{7.14}$$

formundaki Galilean dönüşümü ile deterministik

$$u_t + uu_x + Ru_{xxx} = 0 \tag{7.15}$$

denklemine dönüşür. Bu denklem için sınır koşulları kullanılarak bir çözüm, sınır koşulları kullanılmadan da bir çözüm bulunmuştur. Extended tanh metodu kullanılarak sınır koşulları ile iki çözüm, sınır koşulları olmadan bir çözüm bulunmuştur.

Burgers denklemi için

$$U(X,T) = u(x,t) + W(T), x = X + m(t), t = T, \tag{7.16}$$

$$m(T) = -\int_0^T W(T')dT', \quad W(T) = \int_0^T \eta(T')dT', \tag{7.17}$$

dönüşümü kullanılarak deterministik denklem elde edildikten sonra, sınır koşullarıyla bir, sınır koşulu olmadan bir çözüm bulunmuştur.

Beşinci bölümdeki diğer bir denklem Kuramoto-Sivashinsky denklemi

$$U_t + AUU_X + BU_{XX} + RU_{XXXX} = \eta(T), \tag{7.18}$$

dir. Bu denklem

$$U(X,T) = u(x,t) + W(T), x = X + m(t), t = T, \tag{7.19}$$

$$m(T) = -A\int_0^T W(T')dT', \quad W(T) = \int_0^T \eta(T')dT', \tag{7.20}$$

dönüşümüyle deterministik forma dönüştürülmüş ve sınır koşulları kullanılarak tanh metoduyla 2 çözüm, extended tanh metoduyla da kullanılarak da 2 çözüm elde edilmiştir. Kawahara denklemi için sınır koşulları kullanılarak tanh metoduyla 8 çözüm, extended tanh için ise 4 çözüm bulunmuştur. Son bölüm olan altıncı bölümde Wick-tipli

$$U_t + H_1(t) \diamond U_x + H_2(t) \diamond U \diamond U_x + H_3(t) \diamond U_{xxx} = 0, \tag{7.21}$$

denklemi incelenmiştir. Bu denkleme Hermite dönüşümü uygulanarak

$$\widetilde{U_t} + \widetilde{H_1}(t,z)\widetilde{U_x} + \widetilde{H_2}(t,z)\widetilde{U}\widetilde{U_x} + \widetilde{H_3}(t,z)\widetilde{U_{xxx}} = 0, \tag{7.22}$$

denklemi elde edilir. F-açılım metodu kullanılarak bu denklem için

$$U(t,x) = c_0 - 12A_3 f_0^2 \gamma F^{2\diamond}(\bar{\zeta}), \tag{7.23}$$

çözümü elde edilir.

3. SONUÇ

Galilean ve Hermite dönüşümleri, stokastik denklemlerin deterministik eşdeğerlerini bulmada çok yararlı araçlardır ve fizik, finans gibi farklı alanlarda ortaya çıkan diğer bazı stokastik evolüsyon denklemlerini deterministik eşdeğerlerine dönüştürmek için kullanılabilirler. Bu tezde çalışılan denklemlere singüler bir pertürbasyon eklenerek, stokastik singüler pertürbe denklemler incelenebilir. Bu tür denklemler için çözümlerin incelenmesi sonlu farklar gibi sayısal yöntemlerin kullanılmasını gerektirir.

CURRICULUM VITAE

Mohanad Alaloush was born in 1983 in the Syrian city of Homs. He completed his primary, secondary and secondary education in Homs. In 2003 he joined Al-Baath University, College of Science, Department of Mathematics, and graduated in 2006 in third place. He worked as a lecturer at Al-Baath University and taught several courses to mathematics students and computer engineering students. In 2011 he obtained a master's degree in applied mathematics from Al-Baath University, Institute of Science, Department of Mathematics. In 2015 he left his job and left for Turkey due to the war in Syria. In 2016, he was awarded a scholarship to study PhD at Van Yüzüncü Yıl University by Türkiye Scholarships. Mohanad Alaloush is married and has one child.